CORIBBON HOPF (FACE) ALGEBRAS GENERATED BY LATTICE MODELS

TAKAHIRO HAYASHI

Abstract. By studying "points of the underlying quantum groups" of coquasi-triangular Hopf (face) algebras, we construct ribbon categories for each lattice models without spectral parameter of both vertex and face type. Also, we give a classification of the braiding and the ribbon structure on quantized classical groups and modular tensor categories closely related to quantum $SU(N)_L$-invariants of 3-manifolds.

1. introduction

It is widely accepted that (co-)quasitriangular Hopf algebra is a good algebraic notion which expresses "quantum groups." For, example, each lattice model w of vertex type (and of face type) without spectral parameter naturally generates a coquasitriangular (CQT) Hopf (face) algebra, thanks to the FRT construction and the Hopf closure (or Hopf envelope) construction. The former construction assigns w to the CQT bialgebra (or face algebra) $\mathfrak{A}(w)$ (cf. [30], [24], [31], [7]), while the latter construction assigns some CQT bialgebra (or face algebra) $\mathfrak{H}$ to the CQT Hopf (face) algebra $\mathrm{Hc}(\mathfrak{H})$ ([16], [14]). However, to give applications of CQT Hopf (face) algebras $\mathfrak{H}$ to low-dimensional topology, we need one additional structure on these, which is called the *ribbon functional* on $\mathfrak{H}$, a dual notion of the ribbon element. It is known that there exists a Drinfeld's double of a finite-dimensional Hopf algebra, which has no ribbon element (cf. [22] Proposition 7). Also, it is known that the ribbon functional of a CQT Hopf algebra is not necessarily unique even if it exists. Hence it is natural to investigate sufficient conditions for the existence of the ribbon functional on CQT Hopf (face) algebras, and to develop the classification theory of the ribbon functionals.

One of the purpose of this paper is to prove the existence of the ribbon functionals on CQT Hopf face algebras of the form $\mathrm{Hc}(\mathfrak{A}(\mathrm{w}))$ (cf. Theorem 6.5). This result implies that each w produces a ribbon category, and therefore, it implies that w generates a family of link invariants. We note that when $w = \check{R}$ is a vertex model, this family contains the link invariant constructed by Reshetikhin [29] (see Remark at the end of Section 6). As byproducts, we also obtain several useful results on the ribbon functionals on CQT Hopf face algebras.

The other purpose of this paper is to give the classification of the braidings and the ribbon functionals on the function algebras $\mathrm{Fun}(\mathrm{G_q})$ of the quantized classical groups $G_q = GL_q(N)$, $SL_q(N)$, $SO_q(N)$, $O_q(N)$ and $Sp_q(N)$, and also, on some Hopf face algebras $\mathfrak{S}(A_{N-1};t)_\epsilon$ which are closely related to the $SU(N)_L$-topological quantum field theories. The braiding of these Hopf (face) algebras is not unique. However, the non-uniqueness is explained using certain gradings of these algebras via cyclic groups Γ. The ribbon functionals of these algebras always exist and the number of those is at most two. We note that the proof of the former is very similar to that of the classification of the braidings of $\mathrm{Fun}(\mathrm{Mat_q(N)})$ due to Takeuchi [35], while the proof of the latter essentially depends on our general theory for $\mathrm{Hc}(\mathfrak{A}(\mathrm{w}))$.

Date: Department of Mathematics, Furo-cho, Chikusa-ku, Nagoya 464, Japan.

The paper is organized as follows. In Section 2, we recall basic concepts of the face algebra. The notion of the face algebra generalizes that of the bialgebra, and is necessary to study lattice models of face type and the corresponding link invariants in the framework of the quantum group theory. In Section 3, we recall the relation between lattice models and face algebras. In Section 4, we recall the Hopf closure construction which is the the main tool of this paper. In Section 5, we give a study of group-like elements of the dual algebras $\mathfrak{A}(w)^\circ$ and $\mathrm{Hc}(\mathfrak{H})^\circ$. It plays a crucial role for our study of the ribbon functionals. In Section 6, we give several results on the ribbon functionals of $\mathrm{Hc}(\mathfrak{H})$ and its quotients. In Section 7 and Section 8, we give the results for quantized classical groups and the algebras $\mathfrak{S}(A_{N-1};t)_\epsilon$ stated above.

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Notation. Throughout this paper, we use Sweedler's sigma notation for coalgebras C and their right comodules U, such as $(\Delta\otimes\mathrm{id})(\Delta(\mathrm{a}))=\sum_{(\mathrm{a})}\mathrm{a}_{(1)}\otimes\mathrm{a}_{(2)}\otimes\mathrm{a}_{(3)}$ (cf. [26]). Also, we denote by ρ_U the coaction $U\to U\otimes C$; $u\mapsto\sum_{(u)}u_{(0)}\otimes u_{(1)}$, and by π_U the left action of C^* on U given by $\pi_U(X)u=\sum_{(u)}u_{(0)}X(u_{(1)})$ $(u\in U, X\in C^*)$. For a linear operator A on a vector space W with basis $\{\mathbf{p}\}$, we define its matrix $[A^{\mathbf{p}}_{\mathbf{q}}]_{\mathbf{pq}}$ by $A\mathbf{q}=\sum_{\mathbf{p}}\mathbf{p}\mathbf{A}^{\mathbf{p}}_{\mathbf{q}}$.

2. FACE ALGEBRAS

Let $\mathfrak{H}$ be an algebra over a field $\mathbb{K}$ equipped with a coalgebra structure $(\mathfrak{H},\Delta,\varepsilon)$. Let $\mathcal{V}$ be a finite nonempty set and let $e_{\mathfrak{H},\mathrm{i}}=e_i$ and $\mathring{e}_{\mathfrak{H},\mathrm{i}}=\mathring{e}_i$ $(i\in\mathcal{V})$ be elements of $\mathfrak{H}$. We say that $(\mathfrak{H},\{e_i,\mathring{e}_i\})$ is a $\mathcal{V}$*-face algebra* if the following relations are satisfied:

$$\Delta(ab)=\Delta(a)\Delta(b), \tag{2.1}$$

$$e_ie_j=\delta_{ij}e_i,\quad \mathring{e}_i\mathring{e}_j=\delta_{ij}\mathring{e}_i,\quad e_i\mathring{e}_j=\mathring{e}_je_i, \tag{2.2}$$

$$\sum_{k\in\mathcal{V}}e_k=1=\sum_{k\in\mathcal{V}}\mathring{e}_k, \tag{2.3}$$

$$\Delta(\mathring{e}_ie_j)=\sum_{k\in\mathcal{V}}\mathring{e}_ie_k\otimes\mathring{e}_ke_j,\quad \varepsilon(\mathring{e}_ie_j)=\delta_{ij}, \tag{2.4}$$

$$\varepsilon(ab)=\sum_{k\in\mathcal{V}}\varepsilon(ae_k)\varepsilon(\mathring{e}_kb) \tag{2.5}$$

for each $a,b\in\mathfrak{H}$ and $i,j\in\mathcal{V}$. We call elements e_i and $\mathring{e}_i$ *face idempotents* of $\mathfrak{H}$. We denote by $\mathfrak{E}=\mathfrak{E}_{\mathfrak{H}}$ the subalgebra of $\mathfrak{H}$ generated by face idempotents. It is known that bialgebra is an equivalent notion of $\mathcal{V}$-face algebra with $\#(\mathcal{V})=$. For a $\mathcal{V}$-face algebra, we have the following formulas:

$$\varepsilon(\mathring{e}_ia)=\varepsilon(e_ia),\quad \varepsilon(a\mathring{e}_i)=\varepsilon(ae_i), \tag{2.6}$$

$$\sum_{(a)}a_{(1)}\varepsilon(e_ia_{(2)}e_j)=e_iae_j, \tag{2.7}$$

$$\sum_{(a)}\varepsilon(e_ia_{(1)}e_j)a_{(2)}=\mathring{e}_ia\mathring{e}_j, \tag{2.8}$$

$$\Delta(a)=\sum_{k,l\in\mathcal{V}}\sum_{(a)}e_ka_{(1)}e_l\otimes\mathring{e}_ka_{(2)}\mathring{e}_l, \tag{2.9}$$

$$\sum_{(a)} e_i a_{(1)} e_j \otimes a_{(2)} = \sum_{(a)} a_{(1)} \otimes \mathring{e}_i a_{(2)} \mathring{e}_j, \tag{2.10}$$

$$\Delta(\mathring{e}_i e_j a \mathring{e}_{i'} e_{j'}) = \sum_{(a)} \mathring{e}_i a_{(1)} \mathring{e}_{i'} \otimes e_j a_{(2)} e_{j'} \tag{2.11}$$

for each $a \in \mathfrak{H}$ and $i, j, i', j' \in \mathcal{V}$.

For a $\mathcal{V}$-face algebra $\mathfrak{H}$, its *dual face algebra* $\mathfrak{H}^\circ$ [9] is defined to be the dual coalgebra of $\mathfrak{H}$ equipped with product and face idempotents given by $\langle XY,\, a\rangle = \sum_{(a)} \langle X,\, a_{(1)}\rangle \langle Y,\, a_{(2)}\rangle$ $(X, Y \in \mathfrak{H}^\circ, \mathfrak{a} \in \mathfrak{H})$ and

$$\langle e_{\mathfrak{H}^\circ, i},\, a\rangle = \varepsilon(a e_{\mathfrak{H}, \mathfrak{i}}), \quad \langle \mathring{e}_{\mathfrak{H}^\circ, i},\, a\rangle = \varepsilon(e_{\mathfrak{H}, \mathfrak{i}} a) \quad (a \in \mathfrak{H}, \mathfrak{i} \in \mathcal{V}). \tag{2.12}$$

Let x^+, x^-, e^+ and e^- be elements of an arbitrary algebra A. We say that x^- is an (e^+, e^-)-*generalized inverse* of x^+ if the following four relations are satisfied:

$$x^\mp x^\pm = e^\pm, \quad x^\pm x^\mp x^\pm = x^\pm. \tag{2.13}$$

We note that the (e^+, e^-)-generalized inverse of x^+ is unique if it exists.

We say that a linear map $S : \mathfrak{H} \to \mathfrak{H}$ is an *antipode* of $\mathfrak{H}$, or $\mathfrak{H}$ is a *Hopf* $\mathcal{V}$-*face algebra* if S is the (E^+, E^-)-generalized inverse of $\mathrm{id}_{\mathfrak{H}}$ with respect to the convolution product of $\mathrm{End}_{\mathbb{K}}(\mathfrak{H})$, where

$$E^+(a) = \sum_{k \in \mathcal{V}} \varepsilon(a e_k) e_k, \quad E^-(a) = \sum_{k \in \mathcal{V}} \varepsilon(e_k a) \mathring{e}_k \quad (a \in \mathfrak{H}). \tag{2.14}$$

An antipode of a $\mathcal{V}$-face algebra is an antialgebra-anticoalgebra map, which satisfies

$$S(\mathring{e}_i e_j) = \mathring{e}_j e_i \quad (i, j \in \mathcal{V}). \tag{2.15}$$

Let $\mathfrak{H}$ be a $\mathcal{V}$-face algebra and let $\mathcal{R}^+ = \mathcal{R}^+_{\mathfrak{H}}$ be an element of $(\mathfrak{H} \otimes \mathfrak{H})^*$ with $(m^*(1), (m^{\mathrm{op}})^*(1))$-generalized inverse $\mathcal{R}^- = \mathcal{R}^-_{\mathfrak{H}}$, where $m\colon \mathfrak{H} \otimes \mathfrak{H} \to \mathfrak{H}$ denotes the product of $\mathfrak{H}$. We say that $\mathcal{R}^+$ is a *braiding* of $\mathfrak{H}$ or $(\mathfrak{H}, \mathcal{R}^\pm)$ is a *coquasitriangular* (or *CQT*) $\mathcal{V}$-face algebra if the following relations are satisfied:

$$\mathcal{R}^+ m^*(X) \mathcal{R}^- = (m^{\mathrm{op}})^*(X) \quad (X \in \mathfrak{H}^*), \tag{2.16}$$

$$(m \otimes \mathrm{id})^*(\mathcal{R}^+) = \mathcal{R}^+_{13} \mathcal{R}^+_{23}, \quad (\mathrm{id} \otimes m)^*(\mathcal{R}^+) = \mathcal{R}^+_{13} \mathcal{R}^+_{12}. \tag{2.17}$$

Here for $Z \in (\mathfrak{H} \otimes \mathfrak{H})^*$ and $\{i, j, k\} = \{1, 2, 3\}$, we define $Z_{ij} \in (\mathfrak{H}^{\otimes 3})^*$ by $Z_{ij}(a_1, a_2, a_3) = Z(a_i, a_j)\varepsilon(a_k)$ $(a_1, a_2, a_3 \in \mathfrak{H})$. The braiding $\mathcal{R}^+$ satisfies the following relations:

$$\mathcal{R}^\pm_{12} \mathcal{R}^\pm_{13} \mathcal{R}^\pm_{23} = \mathcal{R}^\pm_{23} \mathcal{R}^\pm_{13} \mathcal{R}^\pm_{12}, \tag{2.18}$$

$$\mathcal{R}^\mp_{\in\ni} \mathcal{R}^\pm_{\infty\in} \mathcal{R}^\pm_{\infty\ni} = \mathcal{R}^\pm_{\infty\ni} \mathcal{R}^\pm_{\infty\in} \mathcal{R}^\mp_{\in\ni}, \quad \mathcal{R}^\mp_{\infty\ni} \mathcal{R}^\mp_{\in\ni} \mathcal{R}^\pm_{\infty\in} = \mathcal{R}^\pm_{\infty\in} \mathcal{R}^\mp_{\in\ni} \mathcal{R}^\mp_{\infty\ni}, \tag{2.19}$$

$$\mathcal{R}^+(\mathring{e}_i e_j a \mathring{e}_k e_l,\, b) = \mathcal{R}^+(a,\, \mathring{e}_j e_l b \mathring{e}_i e_k), \tag{2.20}$$

$$\mathcal{R}^-(\mathring{e}_i e_j a \mathring{e}_k e_l,\, b) = \mathcal{R}^-(a,\, \mathring{e}_k e_i b \mathring{e}_l e_j), \tag{2.21}$$

$$\mathcal{R}^+(\mathring{e}_i e_j,\, a) = \varepsilon(e_j a e_i), \quad \mathcal{R}^+(a,\, \mathring{e}_i e_j) = \varepsilon(e_i a e_j), \tag{2.22}$$

$$\mathcal{R}^-(\mathring{e}_i e_j,\, a) = \varepsilon(e_i a e_j), \quad \mathcal{R}^-(a,\, \mathring{e}_i e_j) = \varepsilon(e_j a e_i) \tag{2.23}$$

for each $a, b \in \mathfrak{H}$ and $i, j, k, l \in \mathcal{V}$. If $\mathfrak{H}$ is a Hopf $\mathcal{V}$-face algebra, then we have:

$$(S \otimes \mathrm{id})^*(\mathcal{R}^+) = \mathcal{R}^-, \quad (\mathrm{id} \otimes S)^*(\mathcal{R}^-) = \mathcal{R}^+. \tag{2.24}$$

Proposition 2.1. *Let $(\mathfrak{H}, \mathcal{R}^{\pm})$ be a CQT $\mathcal{V}$-face algebra.*
(1) *Then, $\mathcal{R}^{-}_{21}\colon a \otimes b \mapsto \mathcal{R}^{-}(b, a)$ gives another braiding of $\mathfrak{H}$.*
(2) *Let Γ be a semigroup and $\mathfrak{H} = \bigoplus_{\gamma \in \Gamma} \mathfrak{H}_\gamma$ a decomposition of $\mathfrak{H}$ such that $\mathfrak{H}_\gamma \mathfrak{H}_\delta \subset \mathfrak{H}_{\gamma\delta}$ and that $\Delta(\mathfrak{H}_\gamma) \subset \mathfrak{H}_\gamma \otimes \mathfrak{H}_\gamma$. Let $\chi\colon \Gamma \times \Gamma \to \mathbb{K}^\times$ be a map such that $\chi(\gamma_1\gamma_2, \delta) = \chi(\gamma_1, \delta)\chi(\gamma_2, \delta)$, $\chi(\gamma, \delta_1\delta_2) = \chi(\gamma, \delta_1)\chi(\gamma, \delta_2)$. Then, there exists a new braiding $\mathcal{R}^{+}_{\chi}$ of $\mathfrak{H}$ given by*

$$\mathcal{R}^{\pm}_{\chi}(a, b) = \chi(\gamma, \delta)^{\pm 1} \mathcal{R}^{\pm}(a, b) \quad (a \in \mathfrak{H}_\gamma, \mathfrak{b} \in \mathfrak{H}_\delta). \tag{2.25}$$

If $(\mathfrak{H}, \mathcal{R}^{\pm})$ is closable, then so is $(\mathfrak{H}, \mathcal{R}^{\pm}_{\chi})$.

Proof. This is straightforward. □

Let $(\mathfrak{H}, \mathcal{R}^{\pm})$ be a CQT Hopf $\mathcal{V}$-face algebra and $\mathcal{V}$ an invertible central element of $\mathfrak{H}^*$. We say that $\mathcal{V}$ is a *ribbon functional* of $\mathfrak{H}$, or $(\mathfrak{H}, \mathcal{V})$ is a *coribbon Hopf $\mathcal{V}$-face algebra* if

$$m^*(\mathcal{V}) = \mathcal{R}^{-}\mathcal{R}^{-}_{21}(\mathcal{V} \otimes \mathcal{V}), \tag{2.26}$$

$$S^*(\mathcal{V}) = \mathcal{V}. \tag{2.27}$$

A map $f : \mathfrak{H} \to \mathfrak{K}$ between $\mathcal{V}$-face algebras is called a *map of $\mathcal{V}$-face algebras* if it is both an algebra and a coalgebra map such that $f(e_i) = e_i$, $f(\mathring{e}_i) = \mathring{e}_i$ for each $i \in \mathcal{V}$. If both $\mathfrak{H}$ and $\mathfrak{K}$ have antipode, then we have

$$f(S(a)) = S(f(a)) \quad (a \in \mathfrak{H}). \tag{2.28}$$

A map $f : \mathfrak{H} \to \mathfrak{K}$ of $\mathcal{V}$-face algebras between CQT $\mathcal{V}$-face algebras is called a *map of CQT $\mathcal{V}$-face algebras* if

$$(f \otimes f)^*(\mathcal{R}^{+}_{\mathfrak{K}}) = \mathcal{R}^{+}_{\mathfrak{H}}. \tag{2.29}$$

An ideal $\mathfrak{I}$ of a $\mathcal{V}$-face algebra $\mathfrak{H}$ is called a *biideal* if it is a coideal of the underlying coalgebra of $\mathfrak{H}$. If in addition, $\mathfrak{H}$ is a CQT $\mathcal{V}$-face algebra and $\mathfrak{I}$ satisfies $\mathcal{R}^{\pm}(\mathfrak{I}, \mathfrak{H}) = \mathcal{R}^{\pm}(\mathfrak{H}, \mathfrak{I}) = \mathfrak{o}$, then $\mathfrak{I}$ is called a *CQT biideal* of $\mathfrak{H}$. For each $\mathcal{V}$-face algebra (resp. CQT $\mathcal{V}$-face algebra) $\mathfrak{H}$ and its biideal (resp. CQT biideal) $\mathfrak{I}$, the quotient $\mathfrak{H}/\mathfrak{I}$ becomes a $\mathcal{V}$-face algebra (resp. CQT $\mathcal{V}$-face algebra) in an obvious manner.

3. LATTICE MODELS AND COMODULES

Let $\mathcal{G}$ be a finite oriented graph with set of vertices $\mathcal{V} = \mathcal{G}^0$. For an edge $\mathbf{p}$, we denote by $\mathfrak{s}(\mathbf{p})$ and $\mathfrak{r}(\mathbf{p})$ its *source* (*start*) and its *range* (*end*) respectively. For each $m \geq 1$, we denote by $\mathcal{G}^m = \coprod_{i,j \in \mathcal{V}} \mathcal{G}^m_{ij}$ the set of *paths* of $\mathcal{G}$ of *length* m, that is, $\mathbf{p} \in \mathcal{G}^{\mathbf{m}}_{\mathbf{ij}}$ if $\mathbf{p}$ is a sequence $(\mathbf{p_1}, \dots, \mathbf{p_m})$ of edges of $\mathcal{G}$ such that $\mathfrak{s}(\mathbf{p}) := \mathfrak{s}(\mathbf{p_1}) = \mathbf{i}$, $\mathfrak{r}(\mathbf{p_n}) = \mathfrak{s}(\mathbf{p_{n+1}})$ $(1 \leq n < m)$ and $\mathfrak{r}(\mathbf{p}) := \mathfrak{r}(\mathbf{p_m}) = \mathbf{j}$. Also, we set $\mathfrak{s}(\mathfrak{i}) = \mathfrak{i} = \mathfrak{r}(\mathfrak{i})$, $\mathcal{G} = \{\}$ and $\mathcal{G}_| = \emptyset$ for each $i \in \mathcal{V}$ and $j \neq i$. Let $\mathfrak{H}(\mathcal{G})$ be the linear span of the symbols $e\binom{\mathbf{p}}{\mathbf{q}}$ $(\mathbf{p}, \mathbf{q} \in \mathcal{G}^{\mathbf{m}}, \mathbf{m} \geq \mathbf{0})$. Then $\mathfrak{H}(\mathcal{G})$ becomes a $\mathcal{V}$-face algebra by setting

$$\mathring{e}_i = \sum_{j \in \mathcal{V}} e\begin{pmatrix} i \\ j \end{pmatrix}, \quad e_j = \sum_{i \in \mathcal{V}} e\begin{pmatrix} i \\ j \end{pmatrix}, \tag{3.1}$$

$$e\begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} e\begin{pmatrix} \mathbf{r} \\ \mathbf{s} \end{pmatrix} = \delta_{\mathfrak{r}(\mathbf{p})\mathfrak{s}(\mathbf{r})}\, \delta_{\mathfrak{r}(\mathbf{q})\mathfrak{s}(\mathbf{s})}\, e\begin{pmatrix} \mathbf{p} \cdot \mathbf{r} \\ \mathbf{q} \cdot \mathbf{s} \end{pmatrix}, \tag{3.2}$$

$$\Delta\left(e\begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix}\right) = \sum_{\mathbf{t} \in \mathcal{G}^m} e\begin{pmatrix} \mathbf{p} \\ \mathbf{t} \end{pmatrix} \otimes e\begin{pmatrix} \mathbf{t} \\ \mathbf{q} \end{pmatrix}, \quad \varepsilon\left(e\begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix}\right) = \delta_{\mathbf{pq}} \tag{3.3}$$

for each $\mathbf{p}, \mathbf{q} \in \mathcal{G}^{\mathbf{m}}$ and $\mathbf{r}, \mathbf{s} \in \mathcal{G}^{\mathbf{n}}$ $(m, n \geq 0)$. Here for paths $\mathbf{p} = (\mathbf{p_1}, \dots, \mathbf{p_m})$ and $\mathbf{r} = (\mathbf{r_1}, \dots, \mathbf{r_n})$, we set $\mathbf{p} \cdot \mathbf{r} = (\mathbf{p_1}, \dots, \mathbf{p_m}, \mathbf{r_1}, \dots, \mathbf{r_n})$ if $\mathfrak{r}(\mathbf{p}) = \mathfrak{s}(\mathbf{r})$ and $m, n \geq 1$, and also, we set $\mathfrak{s}(\mathbf{p}) \cdot \mathbf{p} = \mathbf{p} = \mathbf{p} \cdot \mathfrak{r}(\mathbf{p})$ for each $\mathbf{p} \in \mathcal{G}$ $(m \geq 0)$.

We say that a quadruple $\left(\mathbf{r}\,{}^{\mathbf{p}}_{\mathbf{q}}\,\mathbf{s}\right)$ is a *face* if $\mathbf{p},\mathbf{q},\mathbf{r},\mathbf{s}\in\mathcal{G}^{\mathbf{1}}$ and

$$\mathfrak{s}(\mathbf{p})=\mathfrak{s}(\mathbf{r}),\quad \mathfrak{r}(\mathbf{p})=\mathfrak{s}(\mathbf{s}),\quad \mathfrak{r}(\mathbf{r})=\mathfrak{s}(\mathbf{q}),\quad \mathfrak{r}(\mathbf{q})=\mathfrak{r}(\mathbf{s}). \tag{3.4}$$

We say that $(\mathcal{G},)$ is a *face model* (or $\mathcal{V}$-*face model*) over $\mathbb{K}$ if w is a map which assigns a scalar $w\left[\mathbf{r}\,{}^{\mathbf{p}}_{\mathbf{q}}\,\mathbf{s}\right]\in\mathbb{K}$ to each face $\left(\mathbf{r}\,{}^{\mathbf{p}}_{\mathbf{q}}\,\mathbf{s}\right)$ of $\mathcal{G}$. A face model $(\mathcal{G},)$ is called a *vertex model* if $\#(\mathcal{V})=$. For convenience, we set $w\left[\mathbf{r}\,{}^{\mathbf{p}}_{\mathbf{q}}\,\mathbf{s}\right]=0$ unless $\mathbf{p},\mathbf{q},\mathbf{r},\mathbf{s}\in\mathcal{G}$ satisfy (3.4). For a face model $(\mathcal{G},)$, we identify w with the linear operator on $\mathbb{K}\mathcal{G}:=\bigoplus_{\mathbf{p}\in\mathcal{G}}\mathbb{K}\mathbf{p}$ given by

$$w(\mathbf{p},\mathbf{q})=\sum_{(\mathbf{r},\mathbf{s})\in\mathcal{G}^{\mathbf{2}}}\mathbf{w}\left[\mathbf{r}\,{}^{\mathbf{p}}_{\mathbf{s}}\,\mathbf{q}\right](\mathbf{r},\mathbf{s})\quad((\mathbf{p},\mathbf{q})\in\mathcal{G}). \tag{3.5}$$

A face model is called *invertible* if w is invertible as an operator on $\mathbb{K}\mathcal{G}$. For an invertible face model $(\mathcal{G},)$, we define another face model $(\mathcal{G},^{-})$, using the identification (3.5). An invertible face model is called *star-triangular* (or *Yang-Baxter*) if w satisfies the braid relation $w_1w_2w_1=w_2w_1w_2$, where w_1 and w_2 denote linear operators on $\mathbb{K}\mathcal{G}$ defined by $w_1(\mathbf{p},\mathbf{q},\mathbf{r})=\mathbf{w}(\mathbf{p},\mathbf{q})\otimes\mathbf{r}$ and $w_2(\mathbf{p},\mathbf{q},\mathbf{r})=\mathbf{p}\otimes\mathbf{w}(\mathbf{q},\mathbf{r})$. Here we identify $(\mathbf{p},\mathbf{q},\mathbf{r})\in\mathcal{G}$ with $\mathbf{p}\otimes\mathbf{q}\otimes\mathbf{r}\in(\mathbb{K}\mathcal{G})^{\otimes}$.

For a face model $(\mathcal{G},)$, we define the algebra $\mathfrak{A}(\mathcal{G},)=\mathfrak{A}()$ to be the quotient of $\mathfrak{H}(\mathcal{G})$ modulo the following relations:

$$\sum_{(\mathbf{c},\mathbf{d})\in\mathcal{G}^{\mathbf{2}}}w\left[\mathbf{a}\,{}^{\mathbf{c}}_{\mathbf{b}}\,\mathbf{d}\right]e\begin{pmatrix}\mathbf{c}\cdot\mathbf{d}\\ \mathbf{p}\cdot\mathbf{q}\end{pmatrix}=\sum_{(\mathbf{r},\mathbf{s})\in\mathcal{G}^{\mathbf{2}}}w\left[\mathbf{r}\,{}^{\mathbf{p}}_{\mathbf{s}}\,\mathbf{q}\right]e\begin{pmatrix}\mathbf{a}\cdot\mathbf{b}\\ \mathbf{r}\cdot\mathbf{s}\end{pmatrix}\quad((\mathbf{p},\mathbf{q}),\,(\mathbf{a},\mathbf{b})\in\mathcal{G}^{\mathbf{2}}). \tag{3.6}$$

Then $\mathfrak{A}(w)$ has a unique structure of $\mathcal{V}$-face algebra such that the projection $\mathfrak{H}(\mathcal{G})\to\mathfrak{A}()$ is a map of $\mathcal{V}$-face algebras. For each $n\geq 0$, $\mathbb{K}\mathcal{G}^{\setminus}$ becomes a comodule of $\mathfrak{A}_{\mathfrak{n}}(\mathfrak{w})$ via $\rho(\mathbf{q})=\sum_{\mathbf{p}\in\mathcal{G}^{\setminus}}\mathbf{p}\otimes\mathbf{e}\binom{\mathbf{p}}{\mathbf{q}}$, where the subcoalgebra $\mathfrak{A}_{\mathfrak{n}}(\mathfrak{w})$ of $\mathfrak{A}(w)$ is defined as the linear span of the elements of the form $e\binom{\mathbf{p}}{\mathbf{q}}$ $(\mathbf{p},\mathbf{q}\in\mathcal{G}^{\setminus})$. If $(\mathcal{G},)$ is star-triangular, then there exist unique bilinear pairings $\mathcal{R}^{\pm}$ on $\mathfrak{A}(w)$ such that $(\mathfrak{A}(w),\mathcal{R}^{\pm})$ is a CQT $\mathcal{V}$-face algebra and that

$$\mathcal{R}^{+}\left(e\begin{pmatrix}\mathbf{p}\\ \mathbf{q}\end{pmatrix},\,e\begin{pmatrix}\mathbf{r}\\ \mathbf{s}\end{pmatrix}\right)=w\left[\mathbf{r}\,{}^{\mathbf{q}}_{\mathbf{p}}\,\mathbf{s}\right] \tag{3.7}$$

for each $\mathbf{p},\mathbf{q},\mathbf{r},\mathbf{s}\in\mathcal{G}^{\mathbf{1}}$ (cf. [24], [7], [31], [13]). We call $\mathcal{R}^{+}$ the *canonical braiding* of $\mathfrak{A}(w)$. For a vertex model $w=\check{R}$, $\mathfrak{A}(w)$ coincides with FRT bialgebra A_R, where $R=P\check{R}$ and $P(\mathbf{p},\mathbf{q})=(\mathbf{q},\mathbf{p})$.

Let $\tilde{\mathcal{G}}$ be the orientation-reversed graph of $\mathcal{G}$ and let $\tilde{}:\mathcal{G}\to\tilde{\mathcal{G}}\,;\ \mathbf{p}\mapsto\tilde{\mathbf{p}}$ $(m\geq 0)$ be the canonical bijection which satisfies $\widetilde{\mathbf{p}\cdot\mathbf{q}}=\tilde{\mathbf{q}}\cdot\tilde{\mathbf{p}}$, $\mathfrak{s}(\tilde{\mathbf{p}})=\mathfrak{r}(\mathbf{p})$ and $\mathfrak{r}(\tilde{\mathbf{p}})=\mathfrak{s}(\mathbf{p})$. We also define a new graph $\mathcal{G}_{\mathrm{LD}}$ by setting $\mathcal{G}_{\mathrm{LD}}=\mathcal{V}$ and $\mathcal{G}_{\mathrm{LD}}=\mathcal{G}\coprod\tilde{\mathcal{G}}$. Let $\mathcal{G}\bar{\times}\tilde{\mathcal{G}}$ and $\tilde{\mathcal{G}}\bar{\times}\mathcal{G}$ denote subsets of $\mathcal{G}_{\mathrm{LD}}$ consisting of elements of the form $\mathbf{p}\cdot\tilde{\mathbf{q}}$ and $\tilde{\mathbf{p}}\cdot\mathbf{q}$ $(\mathbf{p},\mathbf{q}\in\mathcal{G})$ respectively. We define linear operators $w_{\mathrm{LD}},w_{\mathrm{LD}}^{-}:\mathbb{K}(\tilde{\mathcal{G}}\bar{\times}\mathcal{G})\to\mathbb{K}(\mathcal{G}\bar{\times}\tilde{\mathcal{G}})$ by

$$w_{\mathrm{LD}}(\tilde{\mathbf{p}}\cdot\mathbf{q})=\sum_{\mathbf{r},\mathbf{s}}\mathbf{w}_{\mathrm{LD}}\left[\mathbf{r}\,{}^{\tilde{\mathbf{p}}}_{\tilde{\mathbf{s}}}\,\mathbf{q}\right]\mathbf{r}\cdot\tilde{\mathbf{s}}\,;\quad \mathbf{w}_{\mathrm{LD}}\left[\mathbf{r}\,{}^{\tilde{\mathbf{p}}}_{\tilde{\mathbf{s}}}\,\mathbf{q}\right]=:\mathbf{w}^{-\mathbf{1}}\left[\mathbf{p}\,{}^{\mathbf{q}}_{\mathbf{r}}\,\mathbf{s}\right], \tag{3.8}$$

$$w_{\mathrm{LD}}^{-}(\tilde{\mathbf{p}}\cdot\mathbf{q})=\sum_{\mathbf{r},\mathbf{s}}\mathbf{w}_{\mathrm{LD}}^{-}\left[\mathbf{r}\,{}^{\tilde{\mathbf{p}}}_{\tilde{\mathbf{s}}}\,\mathbf{q}\right]\mathbf{r}\cdot\tilde{\mathbf{s}}\,;\quad \mathbf{w}_{\mathrm{LD}}^{-}\left[\mathbf{r}\,{}^{\tilde{\mathbf{p}}}_{\tilde{\mathbf{s}}}\,\mathbf{q}\right]=:\mathbf{w}\left[\mathbf{p}\,{}^{\mathbf{q}}_{\mathbf{r}}\,\mathbf{s}\right]. \tag{3.9}$$

We say that a star-triangular $\mathcal{V}$-face model $(\mathcal{G},)$ is *closable* if both w_{LD} and w_{LD}^{-} are invertible. In this case, we define a new $\mathcal{V}$-face model $(\mathcal{G}_{\mathrm{LD}},{}_{\mathrm{LD}})$ by extending w_{LD} on $\mathbb{K}\mathcal{G}_{\mathrm{LD}}$ via $w_{\mathrm{LD}}|_{\mathbb{K}\mathcal{G}}=w$, $w_{\mathrm{LD}}|_{\mathbb{K}(\mathcal{G}\bar{\times}\tilde{\mathcal{G}})}=(w_{\mathrm{LD}}^{-})^{-1}$ and

$$w_{\mathrm{LD}}\left[\tilde{\mathbf{r}}\,{}^{\tilde{\mathbf{p}}}_{\tilde{\mathbf{q}}}\,\tilde{\mathbf{s}}\right]=w\left[\mathbf{s}\,{}^{\mathbf{q}}_{\mathbf{p}}\,\mathbf{r}\right]\quad(\mathbf{p},\mathbf{q},\mathbf{r},\mathbf{s}\in\mathcal{G}). \tag{3.10}$$

We call w_{LD} the *Lyubashenko double* of w. As in case $(\mathcal{G},)$ is a vertex model, $(\mathcal{G}_{\mathrm{LD}}, {}_{\mathrm{LD}})$ is a star-triangular face model.

Let $\mathfrak{H}$ be a $\mathcal{V}$-face algebra and U its (right) comodule. We define its *face space decomposition* $U = \bigoplus_{i,j\in\mathcal{V}} U(i,j)$ by $U(i,j) = \pi_U(\mathring{e}_i e_j)(U)$. Let V be another $\mathfrak{H}$-comodule. We define the *truncated tensor product* $U\overline{\otimes}V$ to be the vector space

$$U\overline{\otimes}V = \bigoplus_{i,j,k\in\mathcal{V}} U(i,k)\otimes V(k,j) \tag{3.11}$$

equipped with the $\mathfrak{H}$-comodule structure given by

$$\rho_{U\bar{\otimes}V}(\overline{u\otimes v}) = \sum_{(u),(v)} \big(u_{(0)}\otimes v_{(0)}\big)\otimes u_{(1)}v_{(1)}, \tag{3.12}$$

where ${}^{-}\colon U\otimes V \to U\bar{\otimes}V$ denotes the projection $\sum_k \pi_U(e_k)\otimes\pi_V(\mathring{e}_k)$. For $\mathfrak{H}$-comodules U, U', V, V' and maps $f\in\mathrm{End}_{\pi(\mathfrak{E})}(\mathrm{U},\mathrm{U}')$, $g\in\mathrm{End}_{\pi(\mathfrak{E})}(\mathrm{V},\mathrm{V}')$, we set

$$f\bar{\otimes}g = (f\otimes g)|_{U\overline{\otimes}V}, \tag{3.13}$$

where $\mathfrak{E} = \mathfrak{E}_{\mathfrak{H}^\circ}$. If both f and g are comodule maps, then so is $f\bar{\otimes}g$. The category $\mathbf{Com}_{\mathfrak{H}}$ of all $\mathfrak{H}$-comodules becomes a monoidal category via $\bar{\otimes}$ and the category $\mathbf{Com}^{\mathbf{f}}_{\mathfrak{H}}$ of all finite-dimensional $\mathfrak{H}$-comodules becomes its sub monoidal category. The category $\mathbf{Com}^{\mathbf{f}}_{\mathfrak{H}}$ is rigid if and only if $\mathfrak{H}$ has a bijective antipode.

Next, suppose $\mathfrak{H}$ has a braiding $\mathcal{R}^{\pm}$. Then $\mathbf{Com}_{\mathfrak{H}}$ and $\mathbf{Com}^{\mathbf{f}}_{\mathfrak{H}}$ become braided categories via the functorial isomorphism $c_{UV}\colon U\overline{\otimes}V \cong V\overline{\otimes}U$ given by

$$c_{UV}(\overline{u\otimes v}) = \sum_{(u),(v)} v_{(0)}\otimes u_{(0)}\mathcal{R}^{+}(u_{(1)}, v_{(1)}). \tag{3.14}$$

If, in addition, $\mathfrak{H}$ has a ribbon functional $\mathcal{V}$, then $\mathbf{Com}^{\mathbf{f}}_{\mathfrak{H}}$ becomes a ribbon category (see e.g. [21]) via twist $\theta_U\colon U\cong U$ given by $\theta_U = \pi_U(\mathcal{V}^{-\infty})$. Conversely, we have the following.

Proposition 3.1 ([24], [21])**.** *Let $\mathfrak{H}$ be a $\mathcal{V}$-face algebra such that either $\mathbf{Com}_{\mathfrak{H}}$ or $\mathbf{Com}^{\mathbf{f}}_{\mathfrak{H}}$ is a braided monoidal category with braiding $\{c_{UV}\}$. Then, $\mathfrak{H}$ becomes a CQT $\mathcal{V}$-face algebra via*

$$\mathcal{R}^{+}(a,b) = \sum_{k,l\in\mathcal{V}} (\varepsilon\otimes\varepsilon)\circ c_{LM}(\mathring{e}_k a e_l\otimes e_l b\mathring{e}_k), \tag{3.15}$$

$$\mathcal{R}^{-}(b,a) = \sum_{k,l\in\mathcal{V}} (\varepsilon\otimes\varepsilon)\circ (c_{ML})^{-1}(\mathring{e}_k a e_l\otimes e_l b\mathring{e}_k), \tag{3.16}$$

where L and M denote arbitrary finite-dimensional sub $\mathfrak{H}$-comodules of $\mathfrak{H}$ such that $\mathring{e}_i a e_j\in L$, $e_j b\mathring{e}_i\in M$ $(i,j\in\mathcal{V})$. If, in addition, $\mathfrak{H}$ has an antipode and $\mathbf{Com}^{\mathbf{f}}_{\mathfrak{H}}$ is a ribbon category with twist $\{\theta_U\}$, then $\mathfrak{H}$ becomes a coribbon Hopf $\mathcal{V}$-face algebra via

$$\mathcal{V}^{\pm\infty}(\dashv) = \varepsilon(\theta^{\mp\infty}_{\mathcal{L}}(\dashv)). \tag{3.17}$$

Proof. To begin with, we note that the existence of such L and M follows from the fundamental theorem of coalgebras, and that (3.15)-(3.17) do not depend on the choice of L and M because of the naturality of c and θ. Here, we will give a proof of the last assertion. Let $\mathcal{V}^{\pm}\in\mathfrak{H}^*$ be as in (3.17) and let U be a finite-dimensional $\mathfrak{H}$-comodule. For each $u^*\in U^*$, we define the $\mathfrak{H}$-comodule map $F_{u^*}: U\to\mathfrak{H}$ by $F_{u^*}(u) = \sum_{(u)}\langle u^*, u_{(0)}\rangle u_{(1)}$ $(u\in U)$. Then, we have

$$\begin{aligned}\langle u^*,\,\theta^{\mp 1}_U(u)\rangle =&\varepsilon\circ F_{u^*}\circ\theta^{\mp1}_U(u) = \varepsilon\circ\theta^{\mp1}_{\mathrm{Im}(\mathrm{F}_{u^*})}\circ F_{u^*}(u)\\ =&\langle u^*,\,\pi_U(\mathcal{V}^{\pm\infty})\sqcap\rangle,\end{aligned} \tag{3.18}$$

or equivalently,

$$\theta^{\mp1}_U(u) = \pi_U(\mathcal{V}^{\pm\infty})\sqcap. \tag{3.19}$$

Rewriting $\langle u^*,\,\theta_U(u)\rangle = \langle\theta_{U^\vee}(u^*), u\rangle$ via this equality, we obtain $S(\mathcal{V}) = \mathcal{V}$.

Let a and b elements of $\mathfrak{H}$ and let L and M be as above. Since $\overline{a \otimes b} = \sum_k ae_k \otimes e_k b$ by (2.5)-(2.7), we have

$$\pi_{L\bar{\otimes}M}(\mathcal{V})(\overline{\dashv \otimes \lfloor}) = \sum_{(\dashv),(\lfloor)} \dashv_{(\infty)} \otimes \lfloor_{(\infty)} \langle \mathcal{V}, \dashv_{(\in)} \lfloor_{(\in)} \rangle \tag{3.20}$$

by (2.11). Using (3.19) and the equality $\theta^{-1}_{L\bar{\otimes}M} = c^{-1}_{L\bar{\otimes}M} \circ c^{-1}_{M\bar{\otimes}L} \circ (\theta_L^{-1} \bar{\otimes} \theta_M^{-1})$, we see that the left-hand side of the above equality is

$$\sum_{(a),(b)} c^{-1}_{L\bar{\otimes}M} \circ c^{-1}_{M\bar{\otimes}L} (\overline{a_{(1)} \otimes b_{(1)}}) \mathcal{V}(\dashv_{(\in)}) \mathcal{V}(\lfloor_{(\in)}) \tag{3.21}$$

$$= \sum_{(a),(b)} a_{(1)} \otimes b_{(1)} \mathcal{R}^-(a_{(2)}, b_{(2)}) \mathcal{R}^-(b_{(3)}, a_{(3)}) \mathcal{V}(\dashv_{(\triangle)}) \mathcal{V}(\lfloor_{(\triangle)}), \tag{3.22}$$

where (3.21) follows from the fact that θ_L and θ_M commute with the action of the face idempotents of $\mathfrak{H}^\circ$. Taking the image via $\varepsilon \otimes \varepsilon$, we get (2.26). $\square$

Let U be a finite-dimensional comodule of a CQT $\mathcal{V}$-face algebra $\mathfrak{H}$. For each $i, j \in \mathcal{V}$, choose a basis $\mathcal{G}_|$ of $U(i,j)$. Let $\mathcal{G}$ be the oriented graph with set of vertexes $\mathcal{V}$ and the set of edges $\mathcal{G} := \coprod_| \mathcal{G}_|$. Then we obtain a star-triangular $\mathcal{V}$-face model $(\mathcal{G}, \mathfrak{u})$ be setting

$$c_{UU}\,(\mathbf{p} \otimes \mathbf{q}) = \sum_{(\mathbf{r},\mathbf{s}) \in \mathcal{G}} w_U\Big[\mathbf{r} \begin{matrix} \mathbf{p} \\ \mathbf{s} \end{matrix} \mathbf{q}\Big]\, \mathbf{r} \otimes \mathbf{s}. \quad ((\mathbf{p}, \mathbf{q}) \in \mathcal{G}). \tag{3.23}$$

4. Drinfeld functionals and the Hopf closure

Let $(\mathfrak{H}, \mathcal{R}^\pm)$ be a CQT $\mathcal{V}$-face algebra. We say that $\mathfrak{H}$ is *closable* (or $\mathfrak{H}$ is a *CCQT* $\mathcal{V}$-face algebra) if there exist both $(\mathcal{F}^+, \mathcal{F}^-)$-generalized inverse $\mathcal{Q}^-$ of $\mathcal{R}^+$ and $(\mathcal{F}^-, \mathcal{F}^+)$-generalized inverse $\mathcal{Q}^+$ of $\mathcal{R}^-$ in the algebra $(\mathfrak{H} \otimes \mathfrak{H}^{\mathrm{cop}})^*$, where $\mathcal{F}^\pm$ denote bilinear forms on $\mathfrak{H}$ defined by

$$\mathcal{F}^+(\dashv, \lfloor) = \sum_{\| \in \mathcal{V}} \varepsilon(\rceil_{\|} \dashv) \varepsilon(\rceil_{\|} \lfloor), \quad \mathcal{F}^-(\dashv, \lfloor) = \sum_{\| \in \mathcal{V}} \varepsilon(\dashv \rceil_{\|}) \varepsilon(\lfloor \rceil_{\|}) \quad (\dashv, \lfloor \in \mathfrak{H}). \tag{4.1}$$

We call $\mathcal{Q}^\pm$ *Lyubashenko forms* of $\mathfrak{H}$. The Lyubashenko forms of a CQT $\mathcal{V}$-face algebra are unique if they exist. If $\mathfrak{H}$ has an antipode, then $\mathfrak{H}$ is closable with Lyubashenko forms given by

$$\mathcal{Q}^+(a,b) = \mathcal{R}^-(S(a), b), \quad \mathcal{Q}^-(a,b) = \mathcal{R}^+(a, S(b)) \quad (a, b \in \mathfrak{H}). \tag{4.2}$$

For a star-triangular face model $(\mathcal{G},)$, $\mathfrak{A}(w)$ is closable if and only if $(\mathcal{G},)$ is closable. In this case, Lyubashenko forms $\mathcal{Q}^\pm$ of $\mathfrak{A}(w)$ satisfy

$$\mathcal{Q}^+\left(e\begin{pmatrix}\mathbf{p}\\ \mathbf{q}\end{pmatrix}, e\begin{pmatrix}\mathbf{r}\\ \mathbf{s}\end{pmatrix}\right) = w^{-1}_{\mathrm{LD}}\Big[\tilde{\mathbf{q}} \begin{matrix}\mathbf{s}\\ \mathbf{r}\end{matrix} \tilde{\mathbf{p}}\Big], \quad \mathcal{Q}^-\left(e\begin{pmatrix}\mathbf{p}\\ \mathbf{q}\end{pmatrix}, e\begin{pmatrix}\mathbf{r}\\ \mathbf{s}\end{pmatrix}\right) = w_{\mathrm{LD}}\Big[\tilde{\mathbf{s}} \begin{matrix}\mathbf{q}\\ \mathbf{p}\end{matrix} \tilde{\mathbf{r}}\Big] \tag{4.3}$$

for each $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s} \in \mathcal{G}^{\mathbf{1}}$.

For a CCQT $\mathcal{V}$-face algebra $\mathfrak{H}$, we define linear functionals $\mathcal{U}_\nu$ $(\nu = 1, 2)$ on $\mathfrak{H}$ via

$$\mathcal{U}_1(a) = \sum_{(a)} \mathcal{Q}^-(a_{(2)}, a_{(1)}), \quad \mathcal{U}_2(a) = \sum_{(a)} \mathcal{Q}^+(a_{(1)}, a_{(2)}) \quad (a \in \mathfrak{H}) \tag{4.4}$$

and call them *Drinfeld functionals* of $\mathfrak{H}$. The Drinfeld functionals of a CCQT $\mathcal{V}$-face algebra $\mathfrak{H}$ are invertible in $\mathfrak{H}^*$ and satisfy the following relations:

$$\mathcal{U}_1^{-1}(a) = \sum_{(a)} \mathcal{Q}^+(a_{(2)}, a_{(1)}), \quad \mathcal{U}_2^{-1}(a) = \sum_{(a)} \mathcal{Q}^-(a_{(1)}, a_{(2)}), \tag{4.5}$$

$$(\mathcal{U}_\nu \otimes \mathcal{U}_\nu)\mathcal{R}^{\pm} = \mathcal{R}^{\pm}(\mathcal{U}_\nu \otimes \mathcal{U}_\nu), \tag{4.6}$$

$$\mathcal{U}_\nu^{\pm 1}(\mathring{e}_i e_j) = \delta_{ij}, \quad \mathcal{U}_\nu^{\pm}(\mathring{e}_i a \mathring{e}_j) = \mathcal{U}_\nu^{\pm}(e_i a e_j), \tag{4.7}$$

$$\mathcal{U}_1\mathcal{U}_2 = \mathcal{U}_2\mathcal{U}_1, \tag{4.8}$$

$$m^*(\mathcal{U}) = \mathcal{R}^-\mathcal{R}_{21}^-(\mathcal{U}\otimes\mathcal{U}) = (\mathcal{U}\otimes\mathcal{U})\mathcal{R}^-\mathcal{R}_{21}^-, \tag{4.9}$$

$$m^*(\mathcal{U}^{-1}) = \mathcal{R}_{21}^+\mathcal{R}^+(\mathcal{U}\otimes\mathcal{U})^{-1} = (\mathcal{U}\otimes\mathcal{U})^{-1}\mathcal{R}_{21}^+\mathcal{R}^+ \tag{4.10}$$

for each $\nu = 1, 2$, $a \in \mathfrak{H}$ and $i, j \in \mathcal{V}$, where $\mathcal{U}$ stands for $\mathcal{U}_\infty$ or $\mathcal{U}_\in^{-\infty}$.

Let $f : \mathfrak{H} \to \mathfrak{K}$ be a map of CQT $\mathcal{V}$-face algebras. If $\mathfrak{K}$ is closable with Lyubashenko forms $\mathcal{Q}_{\mathfrak{K}}^{\pm}$ and Drinfeld functionals $\mathcal{U}_{\nu\,\mathfrak{K}}$, then $\mathfrak{H}$ is also closable with Lyubashenko forms and Drinfeld functionals given by

$$\mathcal{Q}_{\mathfrak{H}}^{\pm} = (f\otimes f)^*(\mathcal{Q}_{\mathfrak{K}}^{\pm}), \quad \mathcal{U}_{\nu\,\mathfrak{H}} = f^*(\mathcal{U}_{\nu\,\mathfrak{K}}). \tag{4.11}$$

Next, we recall the *Hopf closure* (or *Hopf envelope*) construction of CQT Hopf $\mathcal{V}$-face algebras. It is introduced by Phung Ho Hai [16] for bialgebras, and independently, by [14] for face algebras. Let $\mathfrak{H}$ be a CCQT $\mathcal{V}$-face algebra. We denote by $\mathfrak{H}^{\mathrm{bop}}$ its biopposite $\mathcal{V}$-face algebra, that is, $\mathfrak{H}^{\mathrm{bop}}$ is a $\mathcal{V}$-face algebra equipped with the opposite product and the opposite coproduct of $\mathfrak{H}$ together with the face idempotents

$$\mathring{e}_{\mathfrak{H}^{\mathrm{bop}},\mathrm{i}} = e_{\mathfrak{H},\mathrm{i}}, \quad e_{\mathfrak{H}^{\mathrm{bop}},i} = \mathring{e}_{\mathfrak{H},i}. \tag{4.12}$$

Let $\sigma\colon \mathfrak{H} \to \mathfrak{H}^{\mathrm{bop}}$ be the canonical anti-isomorphism, which satisfies

$$\sigma(e_{\mathfrak{H},\mathrm{i}}) = \mathring{e}_{\mathfrak{H}^{\mathrm{bop}},i}, \quad \sigma(\mathring{e}_{\mathfrak{H},\mathrm{i}}) = e_{\mathfrak{H}^{\mathrm{bop}},i} \quad (i\in\mathcal{V}). \tag{4.13}$$

Then

$$\hat{\mathfrak{H}} := \mathfrak{H}\otimes_{\mathfrak{C}}\mathfrak{H}^{\mathrm{bop}} = \bigoplus_{\mathfrak{k},\mathfrak{l}\in\mathcal{V}} \mathfrak{H}\mathring{\mathfrak{e}}_{\mathfrak{k}}\mathfrak{e}_{\mathfrak{l}} \otimes \sigma(\mathfrak{H}\mathring{\mathfrak{e}}_{\mathfrak{l}}\mathfrak{e}_{\mathfrak{k}}) \tag{4.14}$$

becomes a $\mathcal{V}$-face algebra by setting

$$(a\otimes_{\mathfrak{C}}\sigma(b))(c\otimes_{\mathfrak{C}}\sigma(d)) = \sum_{(b),(c)} \mathcal{R}^-(b_{(1)}, c_{(3)})\mathcal{Q}^+(b_{(3)}, c_{(1)})ac_{(2)}\otimes_{\mathfrak{C}}\sigma(db_{(2)}), \tag{4.15}$$

$$\Delta(a\otimes_{\mathfrak{C}}\sigma(b)) = \sum_{(a),(b)} (a_{(1)}\otimes_{\mathfrak{C}}\sigma(b_{(2)}))\otimes(a_{(2)}\otimes_{\mathfrak{C}}\sigma(b_{(1)})), \tag{4.16}$$

$$\varepsilon(a\otimes_{\mathfrak{C}}\sigma(b)) = \sum_{k\in\mathcal{V}} \varepsilon(ae_k)\varepsilon(be_k), \tag{4.17}$$

$$e_{\hat{\mathfrak{H}},i} = e_{\mathfrak{H},\mathrm{i}}\otimes_{\mathfrak{C}}\sigma(1_{\mathfrak{H}}), \quad \mathring{e}_{\hat{\mathfrak{H}},i} = \mathring{e}_{\mathfrak{H},\mathrm{i}}\otimes_{\mathfrak{C}}\sigma(1_{\mathfrak{H}}) \tag{4.18}$$

for each $a, b, c, d \in \mathfrak{H}$ and $i \in \mathcal{V}$, Let $\mathfrak{J}$ be the ideal of $\hat{\mathfrak{H}}$ generated by all elements of the form:

$$\sum_{(a)} (1\otimes_{\mathfrak{C}}\sigma(a_{(1)}))(a_{(2)}\otimes_{\mathfrak{C}}1) - \sum_{k\in\mathcal{V}} \varepsilon(ae_k)e_k, \tag{4.19}$$

$$\sum_{(a)} a_{(1)}\otimes_{\mathfrak{C}}\sigma(a_{(2)}) - \sum_{k\in\mathcal{V}} \varepsilon(e_k a)\mathring{e}_k \quad (a\in\mathfrak{H}). \tag{4.20}$$

It is easy to verify that $\mathfrak{J}$ becomes a biideal. We denote the quotient $\mathcal{V}$-face algebra $\hat{\mathfrak{H}}/\mathfrak{J}$ by $\mathrm{Hc}(\mathfrak{H})$ and call it the *Hopf closure* of $\mathfrak{H}$. For simplicity, we denote an element $a\otimes_{\mathfrak{C}}\sigma(b) + \mathfrak{J}$ of $\mathrm{Hc}(\mathfrak{H})$ by $a\sigma(b)$ for each $a, b \in \mathfrak{H}$. The Hopf closure $\mathrm{Hc}(\mathfrak{H})$ has a unique structure of CQT Hopf $\mathcal{V}$-face algebra such that the canonical map

$\iota\colon \mathfrak{H} \to \mathrm{Hc}(\mathfrak{H})$; $a \mapsto a \otimes_{\mathfrak{C}} 1 + \mathfrak{J}$ $(a \in \mathfrak{H})$ is a map of CQT $\mathcal{V}$-face algebras. Explicitly, the antipode of $\mathrm{Hc}(\mathfrak{H})$ is given by

$$S(a\sigma(b)) = \sum_{(b)} \mathcal{U}_\nu(\lfloor_{(\infty)})\lfloor_{(\in)}\sigma(\dashv)\mathcal{U}_\nu^{-\infty}(\lfloor_{(\ni)}) \quad (\nu = \infty, \in). \tag{4.21}$$

When $\mathfrak{H}$ is a bialgebra, the underlying Hopf algebra of $\mathrm{Hc}(\mathfrak{H})$ agrees with the Hopf envelope of $\mathfrak{H}$ in the sense of Manin [25]. The Hopf closure has the following universal mapping property.

Theorem 4.1. *Let $\mathfrak{H}$ be a CCQT $\mathcal{V}$-face algebra and $\mathfrak{K}$ a CQT Hopf $\mathcal{V}$-face algebra. Let $f : \mathfrak{H} \to \mathfrak{K}$ be a map of CQT $\mathcal{V}$-face algebras. Then there exists a unique map $\bar{f}\colon \mathrm{Hc}(\mathfrak{H}) \to \mathfrak{K}$ of CQT $\mathcal{V}$-face algebras such that $f = \bar{f} \circ \iota$, where $\iota\colon \mathfrak{H} \to \mathrm{Hc}(\mathfrak{H})$ is given by $\iota(a) = a \otimes_{\mathfrak{C}} 1 + \mathfrak{J}$ $(a \in \mathfrak{H})$. Explicitly, we have*

$$\bar{f}(a\sigma(b)) = f(a)S(f(b)). \tag{4.22}$$

Proposition 4.2. *Let $\mathfrak{H}$ be a CQT $\mathcal{V}$-face algebra (resp. CQT Hopf $\mathcal{V}$-face algebra) and U its finite-dimensional comodule. Let $(\mathcal{G}, \mathfrak{w})$ be a face model given by (3.23). Then there exists a unique map $f : \mathfrak{A}(\mathfrak{w}_{\mathfrak{U}}) \to \mathfrak{H}$ (resp. $f : \mathrm{Hc}(\mathfrak{A}(\mathfrak{w}_{\mathfrak{U}})) \to \mathfrak{H}$) of CQT $\mathcal{V}$-face algebras such that $(\mathrm{id}_{\mathbb{K}\mathcal{G}} \otimes \mathrm{f}) \circ \rho_{\mathfrak{H}} = \rho_{\mathfrak{A}(\mathfrak{w}_{\mathfrak{U}})}$ (resp. $(\mathrm{id}_{\mathbb{K}\mathcal{G}} \otimes \mathrm{f}) \circ \rho_{\mathfrak{H}} = \rho_{\mathrm{Hc}(\mathfrak{A}(\mathfrak{w}_{\mathfrak{U}}))}$).*

Proof. See [13] for a proof of the assertion for $\mathfrak{A}(\mathfrak{w}_{\mathfrak{U}})$. The assertion for $\mathrm{Hc}(\mathfrak{A}(\mathfrak{w}_{\mathfrak{U}}))$ follows from that of $\mathfrak{A}(\mathfrak{w}_{\mathfrak{U}})$ and the universal mapping property of Hc. □

Proposition 4.3. *For each CQT Hopf $\mathcal{V}$-face algebra $\mathfrak{H}$, we have:*

$$S^*(\mathcal{U}_1^{\pm 1}) = \mathcal{U}_2^{\mp 1}, \quad S^*(\mathcal{U}_2^{\pm 1}) = \mathcal{U}_1^{\mp 1}, \tag{4.23}$$

$$\mathcal{U}_\nu X \mathcal{U}_\nu^{-1} = (S^2)^*(X) \quad (X \in \mathfrak{H}^*,\ \nu = 1, 2). \tag{4.24}$$

In particular, S is bijective and $\mathcal{U}_1\mathcal{U}_2^{-1}$ is a central element of $\mathfrak{H}^*$.

Proof. (cf. Drinfeld [5]). The relation (4.23) follows from (4.2), (2.24) and (4.5). Substituting $\sum_{(c)} c_{(2)} \otimes S(c_{(1)})$ into $\mathcal{R}^+ m^*(X) = (m^{\mathrm{op}})^*(X)\mathcal{R}^+$, we obtain

$$\begin{aligned}
\sum_{(c)} \mathcal{U}_\infty(\rfloor_{(\in)})\rfloor_{(\ni)}\mathcal{S}(\rfloor_{(\infty)}) &= \sum_{(c)} S(c_{(2)})c_{(3)}\mathcal{Q}^-(c_{(4)}, c_{(1)}) \\
&= \sum_{(c)} \sum_{k \in \mathcal{V}} e_k \mathcal{Q}^-(c_{(2)}\mathring{e}_k, c_{(1)}) \\
&= \sum_{k \in \mathcal{V}} \mathcal{U}_\infty(\mathring{\rceil}_{\|}\rfloor)\rceil_{\|},
\end{aligned}$$

where the second equality follows from (2.8) and the third equality follows from (2.21) and (2.11). Using this relation, we compute

$$\begin{aligned}
\sum_{(c)} S^2(c_{(1)})\mathcal{U}_\infty(\rfloor_{(\in)}) &= \sum_{(c)} \sum_{k \in \mathcal{V}} \mathcal{U}_\infty(\mathring{\rceil}_{\|}\rfloor_{(\in)})\rceil_{\|}\mathcal{S}^{\in}(\rfloor_{(\infty)}) \\
&= \sum_{(c)} \mathcal{U}_\infty(\rfloor_{(\ni)})\rfloor_{(\triangle)}\mathcal{S}(\mathcal{S}(\rfloor_{(\infty)})\rfloor_{(\in)}) \\
&= \sum_{(c)} \sum_{k \in \mathcal{V}} \mathcal{U}_\infty(\rfloor_{(\infty)}\mathring{\rceil}_{\|})\rfloor_{(\in)}\mathring{\rceil}_{\|} \\
&= \sum_{(c)} \mathcal{U}_\infty(\rfloor_{(\infty)})\rfloor_{(\in)},
\end{aligned}$$

where the first equality follows from (2.9) and (2.15) and the last equality follows from (4.7) and (2.9). Substituting this into $X \in \mathfrak{H}^*$, we get $((S^2)^*(X)\mathcal{U}_\infty)(\rfloor) = (\mathcal{U}_\infty\mathcal{X})(\rfloor)$, which proves (4.24) for $\nu = 1$. □

5. Group-like functionals

Let g be an element of a $\mathcal{V}$-face algebra $\mathfrak{H}$. We say that g is *group-like* if

$$\Delta(g) = \sum_{k\in\mathcal{V}} ge_k \otimes g\mathring{e}_k, \tag{5.1}$$

$$g\mathring{e}_i e_j = \mathring{e}_i e_j g, \quad \varepsilon(g\mathring{e}_i e_j) = \delta_{ij} \tag{5.2}$$

for each $i, j \in \mathcal{V}$. We say that a linear functional $\mathcal{G}$ on $\mathfrak{H}$ is *group-like* if it is group-like as an element of the dual face algebra $\mathfrak{H}^\circ$. Explicitly, $\mathcal{G}$ is group-like if and only if it satisfies

$$\mathcal{G}(ab) = \sum_{k\in\mathcal{V}} \mathcal{G}(ae_k)\mathcal{G}(\mathring{e}_k b), \tag{5.3}$$

$$\mathcal{G}(\mathring{e}_i a\mathring{e}_j) = \mathcal{G}(e_i a e_j), \tag{5.4}$$

$$\mathcal{G}(\mathring{e}_i e_j) = \delta_{ij} \tag{5.5}$$

for each $a, b \in \mathfrak{H}$ and $i, j \in \mathcal{V}$. We say that $\mathcal{G}$ is *invertible* if it is invertible as an element of the dual algebra $\mathfrak{H}^*$. We denote by $\mathrm{GLF}(\mathfrak{H})$ the set of all group-like functionals of $\mathfrak{H}$, and by $\mathrm{GLF}(\mathfrak{H})^\times$ the set of all invertible group-like functionals. Note that

$$\mathrm{GLF}(\mathfrak{H}) = \mathrm{Hom}_{\mathbb{K}\text{–}\mathbb{A}\lessdot\eth}(\mathfrak{H}, \mathbb{K}) \tag{5.6}$$

if $\mathfrak{H}$ is a bialgebra.

Lemma 5.1. (1) *The correspondence* $\mathfrak{H} \mapsto \mathrm{GLF}(\mathfrak{H})$ *defines a contravariant functor from the category of* $\mathcal{V}$*-face algebras to the category of semigroups.*
(2) *Let* $\mathfrak{H}$ *be a* $\mathcal{V}$*-face algebra and* $\mathfrak{I}$ *its biideal. Then the projection* $p\colon \mathfrak{H} \to \mathfrak{K} = \mathfrak{H}/\mathfrak{I}$ *gives*

$$p^*\colon \mathrm{GLF}(\mathfrak{K}) \cong \{\mathcal{G} \in \mathrm{GLF}(\mathfrak{H}) \,|\, \mathcal{G}(\mathfrak{I}) = \mathfrak{o}\}. \tag{5.7}$$

(3) *If* $\mathfrak{H}$ *has an antipode, then* $\mathrm{GLF}(\mathfrak{H}) = \mathrm{GLF}(\mathfrak{H})^\times$ *and*

$$S^*(\mathcal{G}) = \mathcal{G}^{-1}. \tag{5.8}$$

for each $\mathcal{G} \in \mathrm{GLF}(\mathfrak{H})$*.*

Proof. The proof of Part (1) is straightforward. Taking the dual of $0 \to \mathfrak{I} \to \mathfrak{H} \to \mathfrak{K} \to \mathfrak{o}$, we obtain

$$p^*\colon \mathfrak{K}^* \cong \{\mathfrak{X} \in \mathfrak{H}^* \,|\, \mathfrak{X}(\mathfrak{I}) = \mathfrak{o}\}. \tag{5.9}$$

It is straightforward to verify that $\mathcal{M} \in \mathrm{GLF}(\mathfrak{K})$ if and only if $p^*(\mathcal{M}) \in \mathrm{GLF}(\mathfrak{H})$ for each $\mathcal{M} \in \mathfrak{K}^*$. This proves Part (2). See [11] Proposition 7.1 for a proof of Part (3). □

Lemma 5.2. *Let* $\mathfrak{H}$ *be a CQT* $\mathcal{V}$*-face algebra.*
(1) *For each group-like functional* $\mathcal{G}$ *on* $\mathfrak{H}$*, we have*

$$(\mathcal{G} \otimes \mathcal{G})\mathcal{R}^\pm = \mathcal{R}^\pm(\mathcal{G} \otimes \mathcal{G}). \tag{5.10}$$

Hence, for each $\mathfrak{H}$*-comodules* U *and* V*, we have*

$$(\pi_V(\mathcal{G})\bar{\otimes}\pi_\mathcal{U}(\mathcal{G}))\rfloor_{\mathcal{U}\mathcal{V}} = \rfloor_{\mathcal{U}\mathcal{V}}(\pi_\mathcal{U}(\mathcal{G})\bar{\otimes}\pi_\mathcal{V}(\mathcal{G})). \tag{5.11}$$

(2) *If* $\mathfrak{H}$ *is closable, then*

$$\mathcal{U}_\infty\mathcal{U}_\in \in \mathrm{GLF}(\mathfrak{H}). \tag{5.12}$$

Proof. Since $\mathcal{R}^+ = (m^{\mathrm{op}})^*(1)\mathcal{R}^+$, we have $(\mathcal{G}\otimes\mathcal{G})\mathcal{R}^+ = (m^{\mathrm{op}})^*(\mathcal{G})\mathcal{R}^+$. Hence the first assertion of Part (1) follows from (2.16). The second assertion follows from the first assertion. Part (2) follows from (4.7) and (4.9)-(4.10). $\square$

Let $\mathfrak{H}$ be a $\mathcal{V}$-face algebra and $\mathcal{G}$ its group-like functional. We define $\mathrm{coad}(\mathcal{G})\colon \mathfrak{H}\to\mathfrak{H}$ by

$$\mathrm{coad}(\mathcal{G})(\dashv) = \sum_{(\dashv)} \mathcal{G}^{-\infty}(\dashv_{(\infty)})\dashv_{(\in)}\mathcal{G}(\dashv_{(\ni)}) \quad (\dashv\in\mathfrak{H}). \tag{5.13}$$

Using (5.3)-(5.5) and (2.10), we see that $\mathrm{coad}(\mathcal{G})$ is an automorphism of $\mathfrak{H}$.

Proposition 5.3. *For $\mathfrak{H}=\mathfrak{H}(\mathcal{G})$ or $\mathfrak{A}(w)$, the map $\mathfrak{H}^*\to\mathrm{End}(\mathbb{K}\mathcal{G})$; $X\mapsto\pi_{\mathbb{K}\mathcal{G}}(X)$ gives the following semigroup isomorphisms*:

$$\mathrm{GLF}(\mathfrak{H}(\mathcal{G}))\cong\mathrm{End}_{\pi(\mathfrak{E})}(\mathbb{K}\mathcal{G}), \tag{5.14}$$

$$\mathrm{GLF}(\mathfrak{A}(\mathrm{w}))\cong\big\{\mathrm{G}\in\mathrm{End}_{\pi(\mathfrak{E})}(\mathbb{K}\mathcal{G})\;\big|\;(\mathcal{G}\bar{\otimes}\mathcal{G})=(\mathcal{G}\bar{\otimes}\mathcal{G})\big\}, \tag{5.15}$$

where $\mathfrak{E}=\mathfrak{E}_{\mathfrak{H}^\circ}$ is as in Sect. 2.

Proof. For each element G of the right-hand side of (5.14), we define a linear functional $\mathcal{G}=\mathcal{G}^{\mathfrak{H}}_{\mathcal{G}}$ on $\mathfrak{H}(\mathcal{G})$ by setting

$$\mathcal{G}\left(\rceil\binom{\rangle}{|}\right)=\delta_{\rangle|},\quad \mathcal{G}\left(\rceil\binom{\mathbf{p}}{\mathbf{q}}\right)=\mathcal{G}^{\mathbf{p_1}}_{\mathbf{q_1}}\cdots\mathcal{G}^{\mathbf{p_m}}_{\mathbf{q_m}} \tag{5.16}$$

for each paths $\mathbf{p}=(\mathbf{p_1},\dots,\mathbf{p_m})$ and $\mathbf{q}=(\mathbf{q_1},\dots,\mathbf{q_m})$ of length $m>0$ and $i,j\in\mathcal{V}$. It is straightforward to verify that $\mathcal{G}$ is a group-like functional of $\mathfrak{H}(\mathcal{G})$. Hence $\pi_{\mathbb{K}\mathcal{G}}$ gives a surjection $\mathrm{GLF}(\mathfrak{H}(\mathcal{G}))\to\mathrm{End}_{\pi(\mathfrak{E})}(\mathbb{K}\mathcal{G})$. Conversely, for $\mathcal{G}\in\mathrm{GLF}(\mathfrak{H}(\mathcal{G}))$, set $G=\pi_{\mathbb{K}\mathcal{G}}(\mathcal{G})$. Then by (5.3)-(5.5), we have (5.16). Thus we get the isomorphism (5.14). Next we show (5.15). By (5.11), $\pi_{\mathbb{K}\mathcal{G}}$ defines a well-defined map from $\mathrm{GLF}(\mathfrak{A}(\mathrm{w}))$ to the right-hand side of (5.15). Hence it suffices to construct the inverse of this map. Let G be an element of the right-hand side of (5.15) and let $\mathcal{G}^{\mathfrak{H}}_{\mathcal{G}}\in\mathrm{GLF}(\mathfrak{H}(\mathcal{G}))$ be as above. By (5.11), we have

$$\mathcal{G}^{\mathfrak{H}}_{\mathcal{G}}\left(\sum_{(\mathbf{c},\mathbf{d})\in\mathcal{G}^{\mathbf{2}}}\sqsupseteq\!\begin{bmatrix}\mathbf{a}\,{}^{\mathbf{c}}_{\mathbf{b}}\,\mathbf{d}\end{bmatrix}\rceil\binom{\mathbf{c}\cdot\mathbf{d}}{\mathbf{p}\cdot\mathbf{q}}-\sum_{(\mathbf{r},\mathbf{s})\in\mathcal{G}^{\mathbf{2}}}\sqsupseteq\!\begin{bmatrix}\mathbf{r}\,{}^{\mathbf{p}}_{\mathbf{s}}\,\mathbf{q}\end{bmatrix}\rceil\binom{\mathbf{a}\cdot\mathbf{b}}{\mathbf{r}\cdot\mathbf{s}}\right)=\prime \tag{5.17}$$

for each $(\mathbf{p},\mathbf{q})$, $(\mathbf{a},\mathbf{b})\in\mathcal{G}^{\mathbf{2}}$. By (5.3), this shows that $\mathcal{G}^{\mathfrak{H}}_{\mathcal{G}}$ vanishes on the biideal $\mathrm{Ker}(\mathfrak{H}(\mathcal{G})\to\mathfrak{A}())$ and that it induces an element of $\mathrm{GLF}(\mathfrak{A}(\mathrm{w}))$. This completes the proof of (5.15). $\square$

Proposition 5.4. *For each CCQT $\mathcal{V}$-face algebra, the canonical map $\iota\colon\mathfrak{H}\to\mathrm{Hc}(\mathfrak{H})$ induces the isomorphism*

$$\iota^*\colon\mathrm{GLF}(\mathrm{Hc}(\mathfrak{H}))\cong\mathrm{GLF}^\times(\mathfrak{H}), \tag{5.18}$$

whose inverse $\mathcal{G}\mapsto\mathcal{G}_{\mathrm{Hc}}$ is given by

$$\mathcal{G}_{\mathrm{Hc}}(\dashv\sigma(\lfloor))=\sum_{||\in\mathcal{V}}\mathcal{G}(\dashv\rceil_{||})\mathcal{G}^{-\infty}(\lfloor\rceil_{||}). \tag{5.19}$$

Proof. By Lemma 5.1 (1), it suffices to show that (5.19) gives the inverse of the correspondence ι^*. It is easy to verify that there exists a linear functional $\hat{\mathcal{G}}\in\hat{\mathfrak{H}}^*$ which sends $a\otimes_{\mathfrak{E}}\sigma(b)$ to the right-hand side of (5.19) and that $\hat{\mathcal{G}}$ satisfies (5.4) and (5.5). Using (5.3) for $\mathcal{G}^{\pm\infty}$, we obtain

$$\hat{\mathcal{G}}((a\otimes_{\mathfrak{E}}1)x(1\otimes_{\mathfrak{E}}\sigma(d)))=\sum_{i,j\in\mathcal{V}}\mathcal{G}(\dashv\rceil_\rangle)\hat{\mathcal{G}}(\mathring{\rceil}_\rangle\S\rceil_|)\mathcal{G}^{-\infty}(\lceil\rceil_|)\quad(\dashv,\lceil\in\mathfrak{H},\mathfrak{x}\in\hat{\mathfrak{H}}). \tag{5.20}$$

By replacing x with $(1\otimes_{\mathfrak{E}}\sigma(b))(c\otimes_{\mathfrak{E}}\sigma(1))$, we obtain

$$\hat{\mathcal{G}}((a\otimes_{\mathfrak{E}}\sigma(b))(c\otimes_{\mathfrak{E}}\sigma(d))) = \sum_{i,j\in\mathcal{V}} \mathcal{G}(\dashv\rceil_{\rangle})\hat{\mathcal{G}}((\infty\otimes_{\mathfrak{E}}\sigma(\lfloor\rceil_{\rangle}))(\rfloor\rceil_{|}\otimes_{\mathfrak{E}}\sigma(\infty)))\mathcal{G}^{-\infty}(\lceil\rceil_{|}). \tag{5.21}$$

On the other hand, using (5.4), (2.10) and (2.21), we obtain

$$\begin{aligned}\hat{\mathcal{G}}((1\otimes_{\mathfrak{E}}\sigma(b))(c\otimes_{\mathfrak{E}}\sigma(1))) &= \sum_{k\in\mathcal{V}}\sum_{(b),(c)} \mathcal{R}^{-}(b_{(1)}e_k, c_{(3)}\mathring{e}_k)\mathcal{Q}^{+}(b_{(3)}, c_{(1)})\mathcal{G}^{-\infty}(\lfloor_{(\in)})\mathcal{G}(\rfloor_{(\in)}) \\ &= \sum_{(b),(c)} \langle (1\otimes\mathcal{G})\mathcal{R}^{-}(\mathcal{G}^{-\infty}\otimes\infty),\ \lfloor_{(\infty)}\otimes\rfloor_{(\in)}\rangle \mathcal{Q}^{+}(\lfloor_{(\in)},\rfloor_{(\infty)}). \end{aligned} \tag{5.22}$$

By (5.10), the right-hand side of the above equality is

$$\sum_{(b),(c)} \mathcal{G}^{-\infty}(\lfloor_{(\infty)})\mathcal{R}^{-}(\lfloor_{(\in)},\rfloor_{(\in)})\mathcal{Q}^{+}(\lfloor_{(\ni)},\rfloor_{(\infty)})\mathcal{G}(\rfloor_{(\ni)}) \;=\; \sum_{\|\in\mathcal{V}} \mathcal{G}^{-\infty}(\rceil_{\|}\lfloor)\mathcal{G}(\rceil_{\|}\rfloor). \tag{5.23}$$

Hence the right-hand side of (5.21) is

$$\sum_{i,j,k\in\mathcal{V}} \mathcal{G}(\dashv\rceil_{\rangle})\mathcal{G}^{-\infty}(\rceil_{\|}\dashv\rceil_{\rangle})\mathcal{G}(\rceil_{\|}\rfloor\rceil_{|})\mathcal{G}^{-\infty}(\lceil\rceil_{|}) = \sum_{\|\in\mathcal{V}} \hat{\mathcal{G}}((\dashv\otimes_{\mathfrak{E}}\sigma(\lfloor))\rceil_{\|})\hat{\mathcal{G}}(\rceil_{\|}(\rfloor\otimes_{\mathfrak{E}}\sigma(\lceil))). \tag{5.24}$$

Thus $\hat{\mathcal{G}}$ is a group-like functional of $\hat{\mathfrak{H}}$. Using (5.3) for $\hat{\mathcal{G}}$, we compute

$$\begin{aligned}&\hat{\mathcal{G}}\left(\mathring{e}_i e_j \left(\sum_{(a)}(1\otimes_{\mathfrak{E}}\sigma(a_{(1)}))(a_{(2)}\otimes_{\mathfrak{E}}1)\right)\mathring{e}_k e_l\right) \\ &= \sum_{m\in\mathcal{V}}\sum_{(a)} \mathcal{G}^{-\infty}(\mathring{\rceil}_{\Updownarrow}\dashv_{(\infty)}\mathring{\rceil}_{|}\rceil_{\rangle})\mathcal{G}(\mathring{\rceil}_{\Updownarrow}\dashv_{(\in)}\mathring{\rceil}_{\|}\rceil_{\updownarrow}) = \sum_{(\dashv)} \delta_{\rangle|}\delta_{\|\updownarrow}\mathcal{G}^{-\infty}(\dashv_{(\infty)}\mathring{\rceil}_{|})\mathcal{G}(\dashv_{(\in)}\rceil_{\updownarrow}) \\ &= \delta_{ij}\delta_{kl}\delta_{jl}\varepsilon(ae_l) = \hat{\mathcal{G}}\left(\mathring{e}_i e_j\left(\sum_{m\in\mathcal{V}}\varepsilon(ae_m)e_m\right)\mathring{e}_k e_l\right) \end{aligned} \tag{5.25}$$

for each $i,j,k,l\in\mathcal{V}$ and $a\in\mathfrak{H}$, where the second equality follows from (5.4) and (2.9) and the third equality follows from (2.11). By repeating similar calculation, we see that $\hat{\mathcal{G}}$ induces a group-like functional $\mathcal{G}_{\mathrm{Hc}}$ on $\mathrm{Hc}(\mathfrak{H})$. Now it is straightforward to verify that $\mathcal{G}\mapsto\mathcal{G}_{\mathrm{Hc}}$ gives the inverse of ι^*. □

Combining Proposition 5.3 and Proposition 5.4, we obtain the group isomorphism

$$\Phi\colon \left\{G\in\mathrm{Aut}_{\pi(\mathfrak{E})}(\mathbb{K}\mathcal{G})\;\middle|\;(\mathcal{G}\bar{\otimes}\mathcal{G}) = (\mathcal{G}\bar{\otimes}\mathcal{G})\right\} \cong \mathrm{GLF}(\mathrm{Hc}(\mathfrak{A}(\mathrm{w}))) \tag{5.26}$$

for each star-triangular face model $(\mathcal{G},)$.

6. A classification theory of ribbon functionals

Lemma 6.1. *For a coribbon Hopf $\mathcal{V}$-face algebra $\mathfrak{H}$, we have*

$$\mathcal{V}^{\pm\infty}(\mathring{\rceil}_{\rangle}\dashv\mathring{\rceil}_{|}) = \mathcal{V}^{\pm\infty}(\rceil_{\rangle}\dashv\rceil_{|}),\quad \mathcal{V}^{\pm\infty}(\mathring{\rceil}_{\rangle}\rceil_{|}) = \delta_{\rangle|}, \tag{6.1}$$

$$\mathcal{V}^{\in} = \mathcal{U}_{\infty}\mathcal{U}_{\in}^{-\infty}, \tag{6.2}$$

$$m^*(\mathcal{V}^{-\infty}) = (\mathcal{V}\otimes\mathcal{V})^{-\infty}\mathcal{R}^{+}_{\in\infty}\mathcal{R}^{+}. \tag{6.3}$$

Proof. The first equality of (6.1) follows from the fact that $\mathcal{V}$ commutes with the face idempotents of $\mathfrak{H}^\circ$. Using (2.26) and (2.23), we obtain

$$\begin{aligned}\mathcal{V}(\dashv) &= \sum_{j,k\in\mathcal{V}}\sum_{(a)}\mathcal{R}^-(e_j,a_{(1)})\mathcal{R}^-(a_{(2)},\mathring{e}_j e_k)\mathcal{V}(\mathring{\rceil}_{\|})\mathcal{V}(\dashv_{(\ni)}) \qquad (6.4)\\ &= \sum_{k\in\mathcal{V}}\mathcal{V}(\mathring{\rceil}_{\|})\mathcal{V}(\mathring{\rceil}_{\|}\dashv)\\ &= \langle\sum_{k\in\mathcal{V}}\mathcal{V}(\mathring{\rceil}_{\|})\mathring{\rceil}_{\|}\mathcal{V},\dashv\rangle,\end{aligned}$$

which implies $\sum_k \mathcal{V}(\mathring{\rceil}_{\|})\mathring{\rceil}_{\|} = \infty$. Since $\{\mathring{e}_k\}$ is linearly independent by the second equality of (2.4), this proves $\mathcal{V}(\mathring{\rceil}_{\|}) = \infty$, or the second equality of (6.1). By a similar discussion to [21] page 351, we obtain $\pi_M(\mathcal{V}^\in) = \pi_M(\mathcal{U}_\infty\mathcal{U}_\in^{-\infty})$ for every $\mathfrak{H}$-comodule M. Hence (6.2) follows from the fundamental theorem of coalgebras (cf. [33] page 46). Using the fact that $\mathcal{R}^-$ is the $(m^*(1),(m^{\mathrm{op}})^*(1))$-generalized inverse of $\mathcal{R}^+$, we obtain

$$\mathcal{R}^-\mathcal{R}^-_{21}\mathcal{R}^+_{21}\mathcal{R}^+ = m^*(1) = \mathcal{R}^+_{21}\mathcal{R}^+\mathcal{R}^-\mathcal{R}^-_{21}. \qquad (6.5)$$

Hence the right-hand side of (6.3) is the inverse of $m^*(\mathcal{V})$ in the algebra $m^*(1)(\mathfrak{H}^{\otimes 2})^*$ $m^*(1)$. This proves (6.3). ☐

Proposition 6.2. *Let $\mathfrak{H}$ be a CQT Hopf $\mathcal{V}$-face algebra and $\mathcal{V}$ an invertible element of $\mathfrak{H}^*$. Then $(\mathfrak{H},\mathcal{V})$ is a coribbon Hopf $\mathcal{V}$-face algebra if and only if $\mathcal{M} = \mathcal{U}_\infty\mathcal{V}^{-\infty}$ is group-like and satisfies the following relations*:

$$\mathcal{M}X\mathcal{M}^{-1} = (S^2)^*(X) \quad (X\in\mathfrak{H}^*), \qquad (6.6)$$

$$\mathcal{M}^\in = \mathcal{U}_\infty\mathcal{U}_\in. \qquad (6.7)$$

Proof. To begin with, we note that the equivalence of $\mathcal{V}\in\mathcal{Z}(\mathfrak{H}^*)$ and (6.6) follows from (4.24), and that that of (2.26) and (5.1) for $g=\mathcal{M}$ follows from (4.9), (6.3) and (6.5). Suppose $\mathcal{V}$ is a ribbon functional. Then the relation (6.7) follows from (6.2) and (4.8), while the first (resp. second) relation of (5.2) for $g=\mathcal{M}$ follows from (4.24) and (2.15) (resp. (4.7) and the second relation of (6.1)). Conversely, if $\mathcal{M}$ satisfies the above conditions, (2.27) follows from (6.2), (4.8) and (5.8). ☐

For a coribbon Hopf $\mathcal{V}$-face algebra $(\mathfrak{H},\mathcal{V})$, we call $\mathcal{M} = \mathcal{U}_\infty\mathcal{V}^{-\infty}$ the *modified ribbon functional* on $\mathfrak{H}$ corresponding to $\mathcal{V}$. For each CQT Hopf $\mathcal{V}$-face algebra $\mathfrak{H}$, we denote by $\mathrm{Rib}(\mathfrak{H})$ the set of all ribbon functionals on $\mathfrak{H}$ and by $\mathrm{MRib}(\mathfrak{H})$ the set of all modified ribbon functionals on $\mathfrak{H}$.

Proposition 6.3. *Let $(\mathfrak{H},\mathcal{R}^\pm)$ be a CQT Hopf $\mathcal{V}$-face algebra.* (1) *We have*

$$\mathrm{MRib}((\mathfrak{H},\mathcal{R}^\mp_{21})) = \mathrm{MRib}((\mathfrak{H},\mathcal{R}^\pm)). \qquad (6.8)$$

(2) *Let $\mathfrak{H}_\gamma$ $(\gamma\in\Gamma)$ and χ be as in Proposition 2.1. Then we have*

$$\mathrm{MRib}((\mathfrak{H},\mathcal{R}^\pm_\chi)) = \mathrm{MRib}((\mathfrak{H},\mathcal{R}^\pm)). \qquad (6.9)$$

Proof. Let $\mathcal{U}_\rangle$, $\mathcal{U}'_\rangle$ and $\mathcal{U}_{\rangle,\chi}$ $(i=1,2)$ be the Drinfeld functionals of $(\mathfrak{H},\mathcal{R}^\pm)$, $(\mathfrak{H},\mathcal{R}^\mp_{21})$ and $(\mathfrak{H},\mathcal{R}^\pm_\chi)$ respectively. Then we have $\mathcal{U}'_\infty = \mathcal{U}_\in$, $\mathcal{U}'_\in = \mathcal{U}_\infty$ and

$$\mathcal{U}_{\infty,\chi}(\dashv) = \chi(\gamma,\gamma)^{-\infty}\mathcal{U}_\infty(\dashv), \quad \mathcal{U}_{\in,\chi}(\dashv) = \chi(\gamma,\gamma)\mathcal{U}_\in(\dashv) \quad (\dashv\in\mathfrak{H}_\gamma). \qquad (6.10)$$

Hence the assertions follows from the definition of the modified ribbon functional and (4.8). ☐

Theorem 6.4. *For each closable star-triangular face model* $(\mathcal{G},)$*, the map* $\pi_{\mathbb{K}\mathcal{G}}$ *gives the following bijection*:

$$\mathrm{Rib}(\mathrm{Hc}(\mathfrak{A}(\mathrm{w}))) \cong \{\mathrm{V} \in \mathrm{Aut}_{\mathrm{Hc}(\mathfrak{A}(\mathrm{w}))}(\mathbb{K}\mathcal{G}) \mid \mathcal{V} = \pi_{\mathbb{K}\mathcal{G}}(\mathcal{U}_\infty \mathcal{U}_\in^{-\infty})\}. \tag{6.11}$$

Equivalently, $\pi_{\mathbb{K}\mathcal{G}}$ *gives*

$$\mathrm{MRib}(\mathrm{Hc}(\mathfrak{A}(\mathrm{w}))) \cong \{\mathrm{M} \mid \mathrm{M}\pi_{\mathbb{K}\mathcal{G}}(\mathcal{U}_\infty)^{-\infty} \in \mathrm{Aut}_{\mathrm{Hc}(\mathfrak{A}(\mathrm{w}))}(\mathbb{K}\mathcal{G}),\, \mathcal{M} = \pi_{\mathbb{K}\mathcal{G}}(\mathcal{U}_\infty \mathcal{U}_\in)\}. \tag{6.12}$$

Proof. Let M be an element of the right-hand side of (6.12). By (4.6), $\pi_{\mathbb{K}\mathcal{G}}(\mathcal{U}_\nu)\bar{\otimes}\pi_{\mathbb{K}\mathcal{G}}(\mathcal{U}_\nu)$ commutes with w for each $\nu = 1, 2$. Hence M belongs to the right-hand side of (5.15). Set $\mathcal{M} = \Phi(\mathcal{M})$, where Φ is as in (5.26). Since $\pi_{\mathbb{K}\mathcal{G}}(\mathcal{M}^\in) = \pi_{\mathbb{K}\mathcal{G}}(\mathcal{U}_\infty \mathcal{U}_\in)$, we have $\mathcal{M}^\in = \mathcal{U}_\infty \mathcal{U}_\in$ by Lemma 5.2 (3). By (4.24), we have $\mathrm{coad}(\mathcal{M})(\rceil\binom{\mathbf{p}}{\mathbf{q}}) = S^{-2}(e\binom{\mathbf{p}}{\mathbf{q}})$, for each $\mathbf{p}, \mathbf{q} \in \mathcal{G}$. Since $\mathrm{coad}(\mathcal{M})$ is an automorphism and $e\binom{\mathbf{p}}{\mathbf{q}}$, $S(e\binom{\mathbf{p}}{\mathbf{q}})$ $(\mathbf{p}, \mathbf{q} \in \mathcal{G})$ and $e\binom{i}{j}$ $(i, j \in \mathcal{V})$ generate $\mathrm{Hc}(\mathfrak{A}(\mathrm{w}))$, this shows that $\mathrm{coad}(\mathcal{M}) = \mathcal{S}^{-\in}$. Thus $\mathcal{M}$ is a modified ribbon functional of $\mathrm{Hc}(\mathfrak{A}(\mathrm{w}))$. Conversely, it is clear that $\pi_{\mathbb{K}\mathcal{G}}$ maps the left-hand side of (6.12) into the right-hand side of (6.12). Thus we get the theorem. □

Theorem 6.5 ([29])**.** *For each closable star-triangular face model* $(\mathcal{G},)$ *over an algebraically closed field* $\mathbb{K}$ *of* $\mathrm{ch}\mathbb{K} \neq \nvDash$, $\mathrm{Hc}(\mathfrak{A}(\mathrm{w}))$ *has a ribbon functional.*

Proof. It suffices to construct a linear operator V which belongs to the right-hand side of (6.11). Let A be the operator $\pi_{\mathbb{K}\mathcal{G}}(\mathcal{U}_\infty \mathcal{U}_\in^{-\infty})$ and $A = S + N$ its Jordan decomposition, that is, S is a diagonalizable operator and N is a nilpotent operator such that $SN = NS$. Let λ_i $(1 \leq i \leq k)$ be (mutually distinct) eigenvalues of S and P_i the projection corresponding to λ_i. It is known that $P_i = f_i(A)$ and $N = g(A)$ for some polynomials $f_i, g \in \mathbb{K}[\mathbb{X}]$. Let $\sqrt{\lambda_i}$ be a square root of λ_i and define a operator V by $V = \sum_i \sqrt{\lambda_i} P_i h(S^{-1}N)$, where $h \in \mathbb{K}[\mathbb{X}]$ is defined by

$$h(X) = 1 + \sum_{n=0}^{\sharp\mathcal{G}} (-1)^n 2^{-2n-1} \frac{1}{n+1} \binom{2n}{n} X^{n+1}. \tag{6.13}$$

Then, we have $V^2 = A$. Since $\mathcal{U}_\infty \mathcal{U}_\in^{-\infty}$ is a central element of $\mathrm{Hc}(\mathfrak{A}(\mathrm{w}))^*$ and V is a polynomial of A, we have $V \in \mathrm{Aut}_{\mathrm{Hc}(\mathfrak{A}(\mathrm{w}))}(\mathbb{K}\mathcal{G})$. By the theorem above, this proves the existence of a ribbon functional on $\mathrm{Hc}(\mathfrak{A}(\mathrm{w}))$. □

Let $(\mathcal{G},)$ be a closable star-triangular face model. We say that $(\mathcal{G},)$ is (*absolutely*) *irreducible* if $\mathbb{K}\mathcal{G}$ is (absolutely) irreducible as an $\mathrm{Hc}(\mathfrak{H})$-comodule. As an immediate consequence of the Theorem 6.4 and Schur's Lemma, we have the following.

Theorem 6.6. *Let* $(\mathcal{G},)$ *be an irreducible closable star-triangular face model over an algebraically closed field. Then we have* $\sharp\mathrm{Rib}(\mathrm{Hc}(\mathfrak{A}(\mathrm{w}))) = 2$ *if* $\mathrm{ch}\mathbb{K} \neq \nvDash$ *and* $\sharp\mathrm{Rib}(\mathrm{Hc}(\mathfrak{A}(\mathrm{w}))) = 1$ *if* $\mathrm{ch}\mathbb{K} = \nvDash$.

Theorem 6.7. *Let* $(\mathcal{G},)$ *be an absolutely irreducible closable star-triangular face model. Suppose* $M \in GL(\mathbb{K}\mathcal{G})$ *satisfies* $\sum_{\mathbf{rs}} M^{\mathbf{p}}_{\mathbf{r}} e\binom{\mathbf{r}}{\mathbf{s}} (M^{-1})^{\mathbf{s}}_{\mathbf{q}} = S^2(e\binom{\mathbf{p}}{\mathbf{q}})$ *and* $\mathrm{Tr}(\mathrm{M}) = \mathrm{Tr}(\mathrm{M}^{-1}) \neq 0$. *Then we have*

$$\mathrm{MRib}(\mathrm{Hc}(\mathfrak{A}(\mathrm{w}))) = \{\Phi(\pm\mathrm{M})\}. \tag{6.14}$$

Proof. By Schur's lemma, we have $\pi(\mathcal{U}_\nu) = c_\nu M$ for some nonzero constant c_ν $(\nu = 1, 2)$. Since $\mathrm{Tr}\pi(\mathcal{U}_\infty) = \mathrm{Tr}\pi(\mathcal{U}_\in^{-\infty})$ by (4.4) and (4.5), we obtain $c_1\mathrm{Tr}(\mathrm{M}) = c_2^{-1}\mathrm{Tr}(\mathrm{M}^{-1})$. Therefore M belongs to the right-hand side of (6.12). □

Proposition 6.8. *Let $(\mathcal{G},)$ be a closable star-triangular face model and let $\mathfrak{K} = \mathrm{Hc}(\mathfrak{A}(\mathrm{w}))/\mathfrak{I}$ be a quotient CQT Hopf $\mathcal{V}$-face algebra of $\mathfrak{H} := \mathrm{Hc}(\mathfrak{A}(\mathrm{w}))$ such that $\mathbb{K}\mathcal{G}$ is absolutely irreducible as a $\mathfrak{K}$-comodule. Then the projection $p\colon \mathfrak{H} \to \mathfrak{K}$ gives the isomorphism*

$$p^*\colon \mathrm{MRib}(\mathfrak{K}) \cong \{\mathcal{M} \in \mathrm{MRib}(\mathrm{Hc}(\mathfrak{A}(\mathrm{w}))) \mid \mathcal{M}(\mathfrak{I}) = \mathfrak{o}\}. \tag{6.15}$$

Proof. We prove the assertion by using Lemma 5.1 (2). Let $\mathcal{M}$ be a group-like functional on $\mathfrak{K}$. It suffices to verify that $\mathrm{coad}(\mathcal{M}) = S^{-2}$ if and only if $\mathrm{coad}(\mathrm{p}^*(\mathcal{M})) = S^{-2}$. Since $\mathrm{coad}(\mathcal{M})(\sqrt{}(\dashv)) = p(\mathrm{coad}(\mathrm{p}^*(\mathcal{M}))(\dashv))$ for each $a \in \mathfrak{H}$, the "if"-part is obvious. Suppose $\mathrm{coad}(\mathcal{M}) = S^{-2}$ and set $M := \pi_{\mathbb{K}\mathcal{G}}(\mathcal{M})$. Since $\pi^{\mathfrak{H}}_{\mathbb{K}\mathcal{G}}(p^*(\mathcal{M})) = \mathcal{M}$, we have $\Phi(M) = p^*(\mathcal{M})$. On the other hand, using (4.11) and Schur's Lemma, we see that $M\pi^{\mathfrak{H}}_{\mathbb{K}\mathcal{G}}(\mathcal{U}_\infty)^{-\infty}$ is a scalar multiple of the identity operator. Hence M belongs to the right-hand side of (6.12). By Theorem 6.4, this proves the proposition. □

Let $\mathfrak{H}$ be a CQT Hopf $\mathcal{V}$-face algebra. We say that $\mathfrak{H}$ is *monogenerated* if there exists an absolutely irreducible $\mathfrak{H}$-comodule U such that $\mathfrak{H}$ is generated by $\mathring{e}_i e_j$ $(i, j \in \mathcal{V})$, the image C of the corepresentation $\mathrm{End(U)}^* \to \mathfrak{H}$ and $S(C)$, as an algebra.

Lemma 6.9. *Let $\mathfrak{H}$ be a CQT Hopf $\mathcal{V}$-face algebra over $\mathbb{K}$ and $\mathbb{F}$ a field extension of $\mathbb{K}$. Then $\mathfrak{H} \otimes \mathbb{F}$ naturally becomes a CQT Hopf $\mathcal{V}$-face algebra over $\mathbb{F}$ and there exists an injection $\mathrm{Rib}(\mathfrak{H}) \to \mathrm{Rib}(\mathfrak{H} \otimes \mathbb{F})$; $\mathcal{V} \mapsto \mathcal{V}_{\mathbb{F}}$ given by $\mathcal{V}_{\mathbb{F}}(\dashv \otimes \infty_{\mathbb{F}}) = \mathcal{V}(\dashv)$.*

Proof. This is straightforward. □

Proposition 6.10. *For each monogenerated CQT Hopf $\mathcal{V}$-face algebra $\mathfrak{H}$, we have $\sharp\mathrm{Rib}(\mathfrak{H}) \leq \in$ if $\mathrm{ch}\mathbb{K} \neq \nvDash$ and $\sharp\mathrm{Rib}(\mathfrak{H}) \leq \infty$ if $\mathrm{ch}\mathbb{K} = \nvDash$*

Proof. Let $(\mathcal{G}, \mathcal{U})$ and $f : \mathrm{Hc}(\mathfrak{A}(\mathfrak{w}_{\mathcal{U}})) \to \mathfrak{H}$ be as in Proposition 4.2. Since $\mathfrak{H}$ is monogenerated, f is surjective for a suitable absolutely irreducible comodule U. Now the assertion is an immediate consequence of (6.15), Theorem 6.6 and the lemma above. □

Remark. (1) To construct a link invariant via a lattice model $(\mathcal{G},)$, it is usual to assume that $(\mathcal{G},)$ is "enhanced" in the sense of [36] (cf. [1], [18], [36]). Theorem 6.5 says that the assumption is superfluous provided that $(\mathcal{G},)$ is closable. For vertex models, this was first proved by Reshetikhin [29].
(2) Combining Theorem 6.5 with the categorical framework of the link invariant [37], we obtain an invariant of framed links colored by comodules of $\mathrm{Hc}(\mathfrak{A}(\mathrm{w}))$, for each closable star-triangular face model $(\mathcal{G},)$. Choosing the $\mathrm{Hc}(\mathfrak{A}(\mathrm{w}))$-comodule $\mathbb{K}\mathcal{G}$ as a color, we obtain an invariant $I_w(L)$ of framed links L which agrees with the known one. However, if $(\mathcal{G},)$ is constructed from a (four-weight) spin model (W_i) ([18], [3]), $I_w(L)$ does not agree with the known invariant $Z_{(W_i)}(L)$. In fact we have $I_w(L) = Z_{(W_i)}(L)Z^*_{(W_i)}(L)$, where $Z^*_{(W_i)}(L)$ is the "dual invariant" of $Z_{(W_i)}(L)$.

7. Quantized classical groups

Let X_l be one of the Dynkin diagram of type A_l, B_l, C_l or D_l, where $l \geq 1$ if $X = A$ and $l \geq 2$ if $X = B, C$ or D. We define integers N and ν by

$$N = \begin{cases} l+1 & (X = A) \\ 2l+1 & (X = B) \\ 2l & (X = C, D), \end{cases} \qquad \nu = \begin{cases} 0 & (X = A) \\ -1 & (X = B, D) \\ 1 & (X = C). \end{cases} \tag{7.1}$$

For $X=B,C,D$ and $1\le i\le N$, we set $i'=N+1-i$ and $\overline{i}=i-\sigma_i\nu/2$, where

$$\sigma_i=\begin{cases}1 & (1\le i<(N+1)/2)\\ 0 & (i=(N+1)/2)\\ -1 & ((N+1)/2<i\le N),\end{cases}\qquad \epsilon_i=\begin{cases}1 & (1\le i\le (N+1)/2)\\ -\nu & ((N+1)/2\le i\le N).\end{cases} \tag{7.2}$$

Also we set $\sigma_i\equiv 1$ for $X=A$. Let $\check{R}=\check{R}_q(X_l)$ be Jimbo's solution of the Yang-Baxter equation of type X_l:

$$\check{R}_q(A_l)=q^{-1}\sum_{r=1}^{N}E_{rr}\otimes E_{rr}+\sum_{r\ne s}E_{rs}\otimes E_{sr}-(q-q^{-1})\sum_{r>s}E_{rr}\otimes E_{ss}, \tag{7.3}$$

$$\check{R}_q(X_l)=\sum_{r;\,r\ne r'}(q^{-1}E_{rr}\otimes E_{rr}+qE_{rr'}\otimes E_{r'r})+\sum_{r;\,r=r'}E_{rr}\otimes E_{rr}+$$
$$\sum_{r,s;\,r\ne s,s'}E_{rs}\otimes E_{sr}+(q-q^{-1})\sum_{r>s}(-E_{rr}\otimes E_{ss}+\epsilon_r\epsilon_s q^{\overline{r}-\overline{s}}E_{rs'}\otimes E_{r's})\quad (X=B,C,D), \tag{7.4}$$

where for $X=A,C,D$ (resp. $X=B$), q (resp. $q^{1/2}$) denotes a non-zero number such that $q^2\ne 1$, and $E_{rs}\in \mathrm{Mat(N,\mathbb{K})}$ denote the matrix units. For $X=B,C,D$, we also set $\lambda=-\nu q^{-N-\nu}$.

For $1\le i,j\le N$, we denote by t_{ij} the element $e\binom{i}{j}$ of $\mathfrak{A}(\check{\mathfrak{R}})$, or its image by an arbitrary bialgebra map. For each $\eta\in\mathbb{K}^\times$, we denote by $\mathcal{R}^+_\eta$ the canonical braiding of the FRT bialgebra $\mathfrak{A}(\eta\check{\mathfrak{R}})$ or its Hopf closure $\mathrm{Hc}(\mathfrak{A}(\eta\check{\mathfrak{R}}))$. Since $\mathfrak{A}(\check{\mathfrak{R}})$ (resp. $\mathrm{Hc}(\mathfrak{A}(\check{\mathfrak{R}}))$) is isomorphic to $\mathfrak{A}(\eta\check{\mathfrak{R}})$ (resp. $\mathrm{Hc}(\mathfrak{A}(\eta\check{\mathfrak{R}}))$) as a bialgebra, we regard $\{\mathcal{R}^+_\eta\}$ as a one-parameter family of braidings of $\mathfrak{A}(\check{\mathfrak{R}})$ (resp. $\mathrm{Hc}(\mathfrak{A}(\check{\mathfrak{R}}))$).

Theorem 7.1 (Takeuchi [35]). *Any braidings of $\mathfrak{A}(\check{\mathfrak{R}}_{\mathfrak{q}}(\mathfrak{X}_{\mathfrak{l}}))$ are either of the form $\mathcal{R}^+_\eta$ or of the form $(\mathcal{R}^-_\eta)_{21}$, where $\eta\in\mathbb{K}^\times$.*

Proof. For $X=A$, this theorem has been proved by M. Takeuchi [35]. Here we give a proof for $X=B,C,D$ by imitating his arguments. It is well known that the operators $g:=\check{R}$ and $e:=(g-g^{-1})/\mu+1$ give a representation of the Birman-Murakami-Wenzl algebra on $(\mathbb{K}^{\mathbb{N}})^{\otimes\not{\mathbb{k}}}$ (cf. [2], [27]), where $\mu=q-q^{-1}$. That is, we have the following formulas:

$$(g_i-\lambda^{-1})(g_i+q)(g_i-q^{-1})=0\quad (i=1,2), \tag{7.5}$$

$$g_1g_2g_1=g_2g_1g_2,\quad e_1g_2e_1=\lambda e_1,\quad e_2g_1e_2=\lambda e_2, \tag{7.6}$$

where, as usual, we set $f_1=f\otimes \mathrm{id}_{\mathbb{K}^{\mathbb{N}}}$ and $f_2=\mathrm{id}_{\mathbb{K}^{\mathbb{N}}}\otimes \mathrm{f}$ for $f\in \mathrm{End}_{\mathbb{K}}((\mathbb{K}^{\mathbb{N}})^{\otimes\not{\mathbb{k}}})$. As consequences of these relations, we also obtain the following formulas:

$$g_i^2=-\mu g_i+\lambda^{-1}\mu e_i+1,\quad e_i^2=\zeta e_i, \tag{7.7}$$

$$e_ig_i=g_ie_i=\lambda^{-1}e_i, \tag{7.8}$$

$$e_ie_je_i=e_i,\quad e_ig_je_i=\lambda e_i, \tag{7.9}$$

$$e_ie_jg_i=e_ig_j-\mu e_ie_j+\mu e_i,\quad g_ie_je_i=g_je_i-\mu e_je_i+\mu e_i, \tag{7.10}$$

$$e_ig_jg_i=e_ie_j,\quad g_ig_je_i=e_je_i, \tag{7.11}$$

$$g_ie_jg_i-g_je_ig_j=\mu(e_ig_j+g_je_i-e_jg_i-g_ie_j)+\mu^2(e_i-e_j) \tag{7.12}$$

for $(i,j)=(1,2),\,(2,1)$, where $\zeta=-(\lambda-\lambda^{-1})\mu^{-1}+1$. By (7.7) and (7.8), we see that $\{g,e,1\}$ is a linear basis of the algebra $\langle g\rangle$.

Let $\mathcal{B}$ be a braiding of $\mathfrak{A}(\check{\mathfrak{R}}))$ and $\check{B}\in\mathrm{End}((\mathbb{K}^{\mathbb{N}})^{\otimes\not{\mathbb{k}}})$ the corresponding solution of the Yang-Baxter equation. Since $\mathfrak{A}_2(\check{\mathfrak{R}})^*$ is the commutant of the algebra $\langle g\rangle$ in

$\mathrm{End}_{\mathbb{K}}((\mathbb{K}^{\mathbb{N}})^{\otimes\not\models})$, $\check{B}$ belongs to the double commutant of $\langle g\rangle$. By [17] page 202, this implies $\check{B} \in \langle g\rangle$. Hence $\check{B}$ is of the form $ag + be + c$ for some $a, b, c \in \mathbb{K}$. Rewriting the Yang-Baxter equation for $\check{B}$ via the formulas above, we obtain

$$\begin{aligned}&(\mu a^2 b + ab^2)X + (-\mu a^2 c + ac^2)Y \\ &\quad + \{b^3 + \mu^2 a^2 b + (\lambda + 2\mu)ab^2 + \lambda^{-1}\mu a^2 c + \zeta b^2 c + bc^2 + 2\lambda^{-1}abc\}Z = 0,\end{aligned} \tag{7.13}$$

where

$$X = e_1 g_2 + g_2 e_1 - e_2 g_1 - g_1 e_2, \quad Y = g_1 - g_2, \quad Z = e_1 - e_2. \tag{7.14}$$

Since X, Y, Z are linearly independent, we obtain three algebraic equations for a, b and c. Solving these, we see that $\check{B}$ is proportional to either g, $g^{-1} = g - \mu e + \mu$, $1 + \alpha e$ or 1, where α denotes a solution of $x^2 + \zeta x + 1 = 0$. Suppose $\check{B} = \eta$ or $\eta(1 + \alpha e)$ for some $\eta \in \mathbb{K}^\times$. Then using (2.17), we obtain

$$\mathcal{B}(\sqcup_{\infty\in}\sqcup_{\in\infty}, \sqcup_{\in\in}) = \eta^{\in}, \quad \mathcal{B}(\sqcup_{\in\infty}\sqcup_{\infty\in}, \sqcup_{\in\in}) = \prime. \tag{7.15}$$

On the other hand, substituting $t_{21} \otimes t_{12}$ into (2.16), we obtain $t_{21}t_{12} = t_{12}t_{21}$, a contradiction. Therefore $\check{B}$ is proportional to either g or g^{-1}. This completes the proof of the theorem. □

The following lemma allows us to apply our general results developed in Sect. 6 to the Hopf closures.

Lemma 7.2. *For each q and η, $\mathbb{K}^{\mathbb{N}} = \mathbb{K}\mathcal{G}$ is absolutely irreducible as a comodule of* $\mathrm{Hc}(\mathfrak{A}(\eta\check{\mathfrak{R}}_{\mathfrak{q}}(\mathfrak{X}_{\mathfrak{l}})))$. *In particular,* $\mathrm{Hc}(\mathfrak{A}(\eta\check{\mathfrak{R}}_{\mathfrak{q}}(\mathfrak{X}_{\mathfrak{l}})))$ *is monogenerated.*

Since the proof of this lemma is quite similar to that of Lemma 7.4 below, we omit it. Next, we determine the ribbon functionals of the Hopf closure of $\mathfrak{A}(\eta\check{\mathfrak{R}}_{\mathfrak{q}}(\mathfrak{X}_{\mathfrak{l}}))$. We note that the following result immediately follows from Theorem 6.7 and the formula (7.37) given below, except for the case $\sum_i q^{2i-N-1-\sigma_i\nu} = 0$.

Proposition 7.3. *For each $\eta \in \mathbb{K}^\times$,* $\mathrm{Hc}(\mathfrak{A}(\eta\check{\mathfrak{R}}_{\mathfrak{q}}(\mathfrak{X}_{\mathfrak{l}})))$ *has exactly two* (*resp. one*) *modified ribbon functionals $\mathcal{M}_\pm$ given by*

$$\mathcal{M}_\pm(\sqcup_{\rangle|}) = \pm\delta_{\rangle|}\amalg^{\in\rangle-\mathcal{N}-\infty-\sigma_\rangle\nu} \tag{7.16}$$

if $\mathrm{ch}\mathbb{K} \neq \not\models$ (*resp.* $\mathrm{ch}\mathbb{K} = \not\models$).

Proof. We will prove this result using Proposition 6.10 and Theorem 6.4. Using (7.37) and (4.24), we obtain

$$(\pi(\mathcal{U}_\infty) \otimes \mathrm{id}) \circ \rho \circ \pi(\mathcal{U}_\infty)^{-\infty}(\sqcap_|) = \sum_\rangle \sqcap_\rangle \otimes \amalg^{\in(\rangle-|)-(\sigma_\rangle-\sigma_|)\nu}\sqcup_{\rangle|} \tag{7.17}$$

$$= (M \otimes \mathrm{id}) \circ \rho \circ \mathrm{M}^{-1}(\mathrm{u_j}), \tag{7.18}$$

where $M := \mathrm{diag}(\mathrm{q}^{2\mathrm{i}-\mathrm{N}-1-\sigma_\mathrm{i}\nu})_\mathrm{i}$. This shows that $M\pi(\mathcal{U}_\infty)^{-\infty}$ commutes with the coaction of $\mathrm{Hc}(\mathfrak{A}(\eta\check{\mathfrak{R}}_{\mathfrak{q}}(\mathfrak{X}_{\mathfrak{l}})))$ on $\mathbb{K}\mathcal{G}$. Hence, it suffices to verify that

$$\pi(\mathcal{U}_\infty\mathcal{U}_\in) = \mathcal{M}^\in. \tag{7.19}$$

By Schur's lemma, we have $\pi(\mathcal{U}_\nu) = \rfloor_\nu\mathcal{M}$ for some constant $c_\nu \in \mathbb{K}^\times$. Suppose $X = B, C$ or D. Using (7.36), we compute

$$\mathcal{U}_\infty(\sqcup_{\infty\infty}) = \sum_{\|=\infty}^{\mathcal{N}} \epsilon_\infty\epsilon_\|\amalg^{\overline{\infty}-\overline{\|}}\eta\check{\mathcal{R}}\left(\|' \,{}^{\infty}_{\|}\, \infty'\right) = \eta\check{\mathcal{R}}\left(\mathcal{N} \,{}^{\infty}_{\infty}\, \mathcal{N}\right) = \eta\amalg. \tag{7.20}$$

Using (4.23) and (7.36), we also obtain

$$\mathcal{U}_\in^{-\infty}(\sqcup_{\mathcal{N}\mathcal{N}}) = \mathcal{U}_\infty(\sqcup_{\infty\infty}) = \eta\amalg. \tag{7.21}$$

This proves $c_1 = \eta q^{N+\nu} = c_2^{-1}$, or (7.19) for $X = B, C, D$. When $X = A$, (7.19) is proved by computing the Lyubashenko double of $\check{R}_q(A_l)$ explicitly. □

Hereafter, we assume that $q^2 \neq -1$ and that $\lambda^{-1} \neq q^{-1}, -q$ if $X = B, C$ or D. By (7.5), this implies

$$\mathbb{K}^\mathrm{N} \otimes \mathbb{K}^\mathrm{N} = \begin{cases} \mathrm{Ker}(\check{\mathrm{R}} - \mathrm{q}^{-1}) \oplus \mathrm{Ker}(\check{\mathrm{R}} + \mathrm{q}) & (X = A) \\ \mathrm{Ker}(\check{\mathrm{R}} - \mathrm{q}^{-1}) \oplus \mathrm{Ker}(\check{\mathrm{R}} + \mathrm{q}) \oplus \mathrm{Ker}(\check{\mathrm{R}} - \lambda^{-1}) & (X = B, C, D) \end{cases} \tag{7.22}$$

as $\mathfrak{A}(\check{\mathfrak{R}})$-comodules. To give the definition of the quantized classical groups, we recall the definition of the (quantum) determinant of $\mathfrak{A}(\check{\mathfrak{R}})$. Let $\Omega = \Omega(X_l)$ be the following q-analogue of the exterior algebra:

$$\Omega(X_l) = \begin{cases} T(\mathbb{K}^\mathrm{N})/(\mathrm{Ker}(\check{\mathrm{R}} - \mathrm{q}^{-1})) & (X = A, C) \\ T(\mathbb{K}^\mathrm{N})/(\mathrm{Im}(\check{\mathrm{R}} + \mathrm{q})) & (X = B, D). \end{cases} \tag{7.23}$$

More explicitly, we have

$$\Omega(A_l) = \big\langle u_i \ (1 \leq i \leq N) \ \big| \ u_i^2 = 0,\, q u_i u_j + u_j u_i = 0 \ (i < j) \big\rangle, \tag{7.24}$$

$$\begin{aligned} \Omega(X_l) = \big\langle u_i \ (1 \leq i \leq N) \ \big| \ & u_i^2 = 0 \ (i \neq (N+1)/2), \\ & q u_i u_j + u_j u_i = 0 \ (i < j, i \neq j'),\ \eta_i = 0 \ (1 \leq i \leq (N+1)/2) \big\rangle \\ & \hspace{6em} (X = B, C, D). \end{aligned} \tag{7.25}$$

Here for $X = B$, C, D and $1 \leq i \leq (N+1)/2$, we set

$$\eta_i = \begin{cases} u_{i'} u_i + u_i u_{i'} - (q - q^{-1}) \sum_{j=1}^{i-1} q^{j-i+1} u_j u_{j'} & (X = B, D,\ i \leq l) \\ u_{l+1} u_{l+1} - (q^{1/2} - q^{-1/2}) \sum_{j=1}^{l} q^{j-l} u_j u_{j'} & (X = B,\ i = l+1) \\ u_{i'} u_i + q^2 u_i u_{i'} + (q - q^{-1}) \sum_{j=i+1}^{l} q^{j-i+1} u_j u_{j'} & (X = C,\ i \leq l). \end{cases} \tag{7.26}$$

Then $\Omega(X_l)$ becomes an $\mathfrak{A}(\check{\mathfrak{R}}_\mathfrak{q}(\mathfrak{X}_\mathfrak{l}))$-comodule algebra via $u_j \mapsto \sum_i u_i \otimes t_{ij}$. For $0 \leq r \leq N$, $\Omega_r := \sum_{i_1,\dots,i_r} \mathbb{K}\approxeq_{\beth_{\nVdash}} \cdots \approxeq_{\beth_{\searrow}}$ is a $\binom{N}{r}$-dimensional subcomodule of Ω. In particular, $\Omega_N = \mathbb{K}\approxeq_{\nVdash}\approxeq_{\nvDash}\dots\approxeq_{\mathbb{N}}$ is one-dimensional and determines the group-like element $\det \in \mathfrak{A}(\check{\mathfrak{R}})$ via the coaction $u_1 \dots u_N \mapsto u_1 \dots u_N \otimes \det$. For $X = B, C, D$, $\mathfrak{A}(\check{\mathfrak{R}})$ has another group-like element quad which is determined by its coaction on the one-dimensional comodule

$$\mathrm{Ker}(\check{\mathrm{R}} - \lambda^{-1}) = \mathbb{K} \sum_{\beth} \epsilon_{\beth\shortparallel} \overline{\beth}^{+\nVdash/\nvDash} \approxeq_{\beth} \otimes \approxeq_{\beth'}. \tag{7.27}$$

By [6] Proposition 5.4-5.5 and the universal mapping property of the Hopf closure and the localization construction, we have

$$\mathrm{Hc}(\mathfrak{A}(\eta\check{\mathfrak{R}}_\mathfrak{q}(\mathfrak{A}_\mathfrak{l}))) \cong \mathfrak{A}(\eta\check{\mathfrak{R}}_\mathfrak{q}(\mathfrak{A}_\mathfrak{l}))[\det^{-1}] =: \mathrm{Fun}\left(\mathrm{GL_q(N)}\right)_\eta, \tag{7.28}$$

$$\mathrm{Hc}(\mathfrak{A}(\eta\check{\mathfrak{R}}_\mathfrak{q}(\mathfrak{X}_\mathfrak{l}))) \cong \mathfrak{A}(\eta\check{\mathfrak{R}}_\mathfrak{q}(\mathfrak{X}_\mathfrak{l}))[\mathrm{quad}^{-1}] \tag{7.29}$$

$$\cong \mathfrak{A}(\eta\check{\mathfrak{R}}_\mathfrak{q}(\mathfrak{X}_\mathfrak{l}))[\det^{-1}] \quad (\mathrm{X = B, C, D}). \tag{7.30}$$

The biideal $(\det -1)$ becomes a CQT biideal of $\mathfrak{A}(\eta\check{\mathfrak{R}})$ if and only if

$$\eta^N = \begin{cases} q & (X = A) \\ 1 & (X = B, C, D), \end{cases} \tag{7.31}$$

while $(\mathrm{quad} - 1)$ becomes a CQT biideal if and only if $\eta = \pm 1$ (cf. [7]).

Now we define the *function algebra of the quantized classical groups* (cf. [34], [6], [7]) to be the CQT bialgebras given by

$$\mathrm{Fun}\left(\mathrm{SL_q(N)}\right)_\eta = \mathfrak{A}(\eta\check{\mathfrak{R}}_\mathfrak{q}(\mathfrak{A}_\mathfrak{l}))/(\det - 1) \quad (\eta^\mathrm{N} = \mathrm{q}), \tag{7.32}$$

$$\mathrm{Fun}\,(\mathrm{SO_q(N)})_\eta = \mathfrak{A}(\eta\check{\mathfrak{R}}_\mathfrak{q}(\mathfrak{X}_\mathfrak{l}))/(\mathrm{det}-1,\mathrm{quad}-1)$$
$$(\eta = 1\,\mathrm{if}\,\mathrm{X}=\mathrm{B}\,\mathrm{and}\,\eta=\pm1\,\mathrm{if}\,\mathrm{X}=\mathrm{D}), \quad (7.33)$$

$$\mathrm{Fun}\,(\mathrm{O_q(N)})_\eta = \mathfrak{A}(\eta\check{\mathfrak{R}}_\mathfrak{q}(\mathfrak{X}_\mathfrak{l}))/(\mathrm{quad}-1) \quad (\eta=\pm1,\mathrm{X}=\mathrm{B},\mathrm{D}) \quad (7.34)$$

$$\mathrm{Fun}\,(\mathrm{Sp_q(N)})_\eta = \mathfrak{A}(\eta\check{\mathfrak{R}}_\mathfrak{q}(\mathfrak{C}_\mathfrak{l}))/(\mathrm{quad}-1) \quad (\eta=\pm1). \quad (7.35)$$

See [6] for a justification of these definitions in case $\mathbb{K}=\mathbb{C}$ and q is transcendental over $\mathbb{Q}$. For $G_q = SL_q(N), SO_q(N)$, etc., we denote by $\mathrm{Fun(G_q)}$ the underlying bialgebra of $\mathrm{Fun(G_q)}_\eta$, and by $\mathcal{R}^+_{\eta,G_q}$ the braiding of $\mathrm{Fun(G_q)}_\eta$. Each of these algebras has an antipode. For example, the antipode of the algebras given in (7.33)-(7.35) is given by

$$S(t_{ij}) = \epsilon_i\epsilon_j\, q^{\overline{i}-\overline{j}}\, t_{j'i'}. \quad (7.36)$$

The square of the antipode of the algebras given in (7.28), (7.29), (7.32)-(7.35) is given by

$$S^2(t_{ij}) = q^{2(i-j)-(\sigma_i-\sigma_j)\nu}\, t_{ij}. \quad (7.37)$$

Lemma 7.4. *Let F be either* $\mathrm{Hc}(\mathfrak{A}(\check{\mathfrak{R}}))$ *or one of the algebras given in* (7.32)-(7.35)*. Then each of the F-comodules* $\mathbb{K}^\mathbb{N}$, $\mathrm{Ker}(\check{\mathrm{R}}-\mathrm{q}^{-1})$ *and* $\mathrm{Ker}(\check{\mathrm{R}}+\mathrm{q})$ *are absolutely irreducible. In particular, F is monogenerated.*

Proof. Since $\mathbb{K}$ is arbitrary, it suffices to show the irreducibility of these comodules. Here we give a proof for $W := \mathrm{Ker}(\check{\mathrm{R}}_\mathrm{q}(\mathrm{C_l})+\mathrm{q})$. To simplify the computation, it is convenient to identify W with its image via the projection $(\mathbb{K}^\mathbb{N})^{\otimes\not\models} \to \Omega_{\not\models}$.

Following [30], we define $K_i, E_i, F_i \in F^*$ $(1 \le i \le l)$ by

$$K_i = \mathcal{R}^-_\eta(t_{ii},-), \quad (7.38)$$

$$E_i = \begin{cases} -\eta^{-1}(q-q^{-1})^{-1}\mathcal{R}^+_\eta(-,\,t_{i+1\;i}) & (1\le i<l) \\ -\eta^{-1}q^{-1}(q^2-q^{-2})^{-1}\mathcal{R}^+_\eta(-,\,t_{l+1\;l}) & (i=l), \end{cases} \quad (7.39)$$

$$F_i = \begin{cases} \eta(q-q^{-1})^{-1}\mathcal{R}^-_\eta(t_{i\;i+1},\,-) & (1\le i<l) \\ \eta q(q^2-q^{-2})^{-1}\mathcal{R}^-_\eta(t_{l\;l+1},\,-) & (i=l). \end{cases} \quad (7.40)$$

Then these elements belong to the dual Hopf algebra F° (cf. [24]) and satisfy

$$\pi_{\mathbb{K}^\mathbb{N}}(K_i) = \eta^{-1}\sum_{k=1}^{N} q^{\delta_{ki}-\delta_{ki'}}E_{kk}, \quad (7.41)$$

$$\pi_{\mathbb{K}^\mathbb{N}}(E_i) = E_{i\,i+1} - qE_{(i+1)'\,i'} \quad (i<j), \quad \pi_{\mathbb{K}^\mathbb{N}}(E_l) = E_{l\;l+1}, \quad (7.42)$$

$$\pi_{\mathbb{K}^\mathbb{N}}(F_i) = E_{i+1\,i} - q^{-1}E_{i'\,(i+1)'} \quad (i<j), \quad \pi_{\mathbb{K}^\mathbb{N}}(F_l) = E_{l+1\;l}, \quad (7.43)$$

$$\Delta(K_i) = K_i \otimes K_i, \quad (7.44)$$

$$\Delta(E_i) = E_i\otimes K_i^{-1} + K_{i+1}^{-1}\otimes E_i, \quad \Delta(F_i) = F_i\otimes K_{i+1} + K_i\otimes F_i, \quad (7.45)$$

where K_{l+1} is given by (7.38).

As a $\langle K_i\rangle$-module, W is the direct sum of the mutually non-isomorphic, non-trivial comodules $\mathbb{K}\approx_{\sqsupset}\approx_{\sqsupset}$ $(j\ne i,i')$ and the trivial comodule $T=\bigoplus_{i=1}^{l-1}\mathbb{K}(\shortparallel\approx_{\sqsupset}\approx_{\sqsupset'} - \approx_{\sqsupset+\not\Vdash}\approx_{(\sqsupset+\not\Vdash)'})$. Hence any non-zero subcomodule M of W contains a vector $v\ne$

0 which belongs to one of these $\langle K_i\rangle$-modules. By verifying $T \cap (\bigcap_i \mathrm{Ker}\pi(\mathrm{E_i}))$ $= 0$, we see that $u_1u_2 \in \mathbb{KE}_{\beth_{\nVdash}} \cdots \mathbb{E}_{\beth_\neg}\gtrapprox$ for some $i_1, \dots, i_k$. Also, by verifying $T = \sum_{i=1}^{l-1} \mathbb{KF}_{\beth}(\approxeq_{\beth}\approxeq_{(\beth+\nVdash)'})$, we see that u_1u_2 generates W as an $\langle F_i\rangle$-module. Thus, W is irreducible as a $\langle K_i, E_i, F_i\rangle$-module, and also, it is irreducible as an F-comodule. ◻

Theorem 7.5. (1) *Any braidings of* $\mathrm{Hc}(\mathfrak{A}(\check{\mathfrak{R}}_{\mathfrak{q}}(\mathfrak{X}_{\mathfrak{l}})))$ *are either of the form* $\mathcal{R}^+_\eta$ *or of the form* $(\mathcal{R}^-_\eta)_{21}$*, where* $\eta \in \mathbb{K}^\times$*.*
(2) *Let* G_q *be either* $SL_q(N)$*,* $SO_q(N)$*,* $O_q(N)$ *or* $Sp_q(N)$*. Then, any braiding of* $\mathrm{Fun(G_q)}$ *is either of the form* $\mathcal{R}^+_{\eta,G_q}$ *or of the form* $(\mathcal{R}^-_{\eta,G_q})_{21}$*, where* η *is as in* (7.32)-(7.35).

Proof. Let $\check{B}$ be a solution of the Yang-Baxter equation, which corresponds to a braiding of one of the above algebras F. By Lemma 7.4, $\mathrm{End_F}((\mathbb{K}^{\mathrm{N}})^{\otimes\nvDash})$ is spanned by two or three projections onto eigenspaces of $\check{R}$, according to $X = A$ or $X = B, C, D$. By linear algebra, these projections are polynomials of $\check{R}$. Therefore, we have $\mathrm{End_F}((\mathbb{K}^{\mathrm{N}})^{\otimes\nvDash}) = \langle\check{R}\rangle$. Hence, by the discussions in the proof of Theorem 7.1, we see that $\check{B}$ is proportional to either $\check{R}$ or $\check{R}^{-1}$. Thus this theorem follows from the result of [7] stated above. ◻

Let F be one of the Hopf algebras given in (7.32)-(7.35). We define the cyclic group $\Gamma = \Gamma_F$ as follows:

$$\Gamma = \begin{cases} \mathbb{Z}/\mathbb{N}\mathbb{Z} & (G_q = SL_q(N)) \\ \mathbb{Z}/\nvDash\mathbb{Z} & (G_q = O_q(N),\, SO_q(2l),\, Sp_q(N)) \\ \{1\} & (G_q = SO_q(2l+1)). \end{cases} \tag{7.46}$$

For $F = \mathfrak{A}(\check{\mathfrak{R}})$ and $\mathrm{Hc}(\mathfrak{A}(\check{\mathfrak{R}}))$, we also set $\Gamma_F = \mathbb{Z}$. Since $\det \in \mathfrak{A}_{\mathfrak{N}}(\check{\mathfrak{R}})$ and $\mathrm{quad} \in \mathfrak{A}_2(\check{\mathfrak{R}})$, the grading $\mathfrak{A}(\check{\mathfrak{R}}) = \bigoplus_n \mathfrak{A}_{\mathfrak{n}}(\check{\mathfrak{R}})$ naturally induces a Γ-grading of F satisfying the properties stated in Proposition 2.1 (2). Now we can restate our classification theorems for braidings as follows.

Corollary 7.6. *Let* F *be one of the bialgebras treated in Theorem 7.1 and Theorem 7.5 and let* Γ *be the cyclic group defined as above. Then, any braiding of* F *is either of the form* $\mathcal{R}^+_\chi$ *or of the form* $(\mathcal{R}^-_\chi)_{21}$*, where* χ *is as in Proposition 2.1. In particular,* $\mathrm{MRib(F)}$ *does not depend on the choice of the braiding of* F (*cf. Proposition 6.3*).

Next, we give the classification theorem of the ribbon functionals for the algebras given in (7.32)-(7.35).

Lemma 7.7. *Let* $\mathcal{M}_\pm$ *be as in* (7.16)*. Then we have*

$$\mathcal{M}_\pm(\det) = (\pm 1)^{\mathrm{N}}, \quad \mathcal{M}_\pm(\mathrm{quad}) = 1. \tag{7.47}$$

Proof. We calculate

$$\begin{aligned} \mathcal{M}_\pm(\det)\mathrm{u_1}\cdots\mathrm{u_N} &= \pi_{\Omega_{\mathrm{N}}}(\mathcal{M}_\pm)(\sqcap_\infty \cdots \sqcap_{\mathcal{N}}) = (\mathcal{M}_\pm\sqcap_\infty)\cdots(\mathcal{M}_\pm\sqcap_{\mathcal{N}}) \\ &= \prod_i \left(\pm q^{2i-N-1-\sigma_i\nu}\right) u_1\cdots u_N = (\pm 1)^N u_1\cdots u_N. \end{aligned} \tag{7.48}$$

The proof of the second formula is similar. ◻

In view of the universal mapping property of the Hopf closure, we see that we may replace $\mathfrak{A}(\eta\check{\mathfrak{R}})$ in (7.32)-(7.35) with $\mathrm{Hc}(\mathfrak{A}(\eta\check{\mathfrak{R}}))$. Hence, as an immediate consequence of Proposition 6.8 and the lemma above, we obtain the following.

Theorem 7.8. (1) *Let G_q be either $SL_q(N)$, $SO_q(N)$ or $Sp_q(N)$, and let p : $\mathrm{Hc}(\mathfrak{A}(\eta\check{\mathfrak{R}}_{\mathrm{q}}(\mathfrak{X}_{\mathfrak{l}}))) \to \mathrm{Fun}(\mathrm{G_q})_\eta$ denote the projection. Then we have*

$$\mathrm{MRib}\,(\mathrm{Fun}(\mathrm{G_q})_\eta) = \begin{cases} \{\mathcal{M}_+ \circ \sqrt{}, \mathcal{M}_- \circ \sqrt{}\} & (N \in 2\mathbb{Z}) \\ \{\mathcal{M}_+ \circ \sqrt{}\} & (N \in 1 + 2\mathbb{Z}), \end{cases} \tag{7.49}$$

where η is as in (7.32)-(7.35).
(2) *We have*

$$\mathrm{MRib}\left(\mathrm{Fun}\,(\mathrm{O_q(N)})_{\pm 1}\right) = \{\mathcal{M}_+ \circ \nabla, \mathcal{M}_- \circ \nabla\}, \tag{7.50}$$

where $r \colon \mathrm{Hc}(\mathfrak{A}(\pm\check{\mathfrak{R}}_{\mathrm{q}}(\mathfrak{X}_{\mathfrak{l}}))) \to \mathrm{Fun}\,(\mathrm{O_q(N)})_{\pm 1}$ denotes the projection.

8. SOS algebras

Let $N \geq 2$ and $L \geq 2$ be integers. Let $\mathcal{C}$ be an $\mathbb{C}$-abelian semisimple rigid monoidal category whose simple objects L_λ are indexed by the following set of partitions:

$$\mathcal{V} = \mathcal{V}_{\mathcal{NL}} := \{\lambda = (\lambda, \dots, \lambda_{\mathcal{N}}) \in \mathbb{Z}^{\mathbb{N}} \mid \mathbb{L} \geq \lambda_{\not{\mathbb{K}}} \geq \cdots \geq \lambda_{\mathbb{N}} = \not{\mathbb{K}}\}. \tag{8.1}$$

We say that $\mathcal{C}$ is an *$SU(N)_L$-category* if the structure constants of its Grothendiek ring agree with the fusion rules $N_{\lambda\mu}^{\nu}$ of $SU(N)_L$-WZW models. The $SU(N)_L$-categories play crucial roles to construct $SU(N)_L$-topological quantum field theories, or corresponding invariants of 3-manifolds (cf. [37]). It is known that two $SU(N)_L$-categories are equivalent to each other up to a “twist” of the associativity constraint (cf. Kazhdan-Wenzl [23]).

In [15], we have constructed a coribbon Hopf $\mathcal{V}_{\mathcal{NL}}$-face algebra $\mathfrak{S} = \mathfrak{S}(\mathfrak{A}_{\mathfrak{N}-\mathfrak{1}}; \mathfrak{t})_\epsilon$ such that $\mathbf{Com}^{\mathbf{f}}_{\mathfrak{S}}$ is an $SU(N)_L$-category. In this section, we determine the braiding and the ribbon structure of $\mathbf{Com}^{\mathbf{f}}_{\mathfrak{S}}$ or equivalently, those of $\mathfrak{S}$ (cf. Proposition 3.1).

To begin with, we recall the definition of $SU(N)_L$-SOS model. For each $1 \leq i \leq N$, we set $\hat{i} = (\delta_{1i}, \dots, \delta_{Ni}) \in \mathbb{Z}^{\mathbb{N}}$. For $m \geq 0$, we define the subset $\mathcal{G}^m$ of $\mathcal{V}^{m+1}$ by

$$\mathcal{G}^m = \mathcal{V}^{m+1} \cap \{\mathbf{p} = (\lambda \,|\, \mathbf{i_1}, \dots, \mathbf{i_m}) \mid \lambda \in \mathcal{V},\ \leq, \dots,\ \leq \mathcal{N}\}, \tag{8.2}$$

where for $\lambda \in \mathbb{Z}^{\mathbb{N}}$ and $1 \leq i_1, \dots, i_m \leq N$, we set

$$(\lambda \,|\, i_1, \dots, i_m) = (\lambda, \lambda + \hat{i}_1, \dots, \lambda + \hat{i}_1 + \cdots + \hat{i}_m), \tag{8.3}$$

and we identify $(\lambda_1 + 1, \cdots, \lambda_N + 1) \in \mathbb{Z}^{\mathbb{N}}$ with $\lambda \in \mathcal{V}$. Then $(\mathcal{V}, \mathcal{G})$ defines an oriented graph $\mathcal{G} = \mathcal{G}_{\mathcal{N},\mathcal{L}}$ and $\mathcal{G}^m$ is identified with the set of paths of $\mathcal{G}$ of length m. For $\mathbf{p} = (\lambda \,|\, \mathbf{i}, \mathbf{j})$, we set $\mathbf{p}^\dagger = (\lambda \,|\, j, i)$ and $d(\mathbf{p}) = \lambda_{\mathbf{i}} - \lambda_{\mathbf{j}} + \mathbf{j} - \mathbf{i}$. We define subsets $\mathcal{G}^2[\rightarrow]$, $\mathcal{G}^2[\downarrow]$ and $\mathcal{G}^2[\searrow]$ of $\mathcal{G}^2$ by

$$\mathcal{G}^2[\rightarrow] = \{\mathbf{p} \in \mathcal{G}^{\mathbf{2}} \mid \mathbf{p}^\dagger = \mathbf{p}\}, \quad \mathcal{G}^{\mathbf{2}}[\downarrow] = \{\mathbf{p} \in \mathcal{G}^{\mathbf{2}} \mid \mathbf{p}^\dagger \notin \mathcal{G}\}, \tag{8.4}$$

$$\mathcal{G}^2[\searrow] = \{\mathbf{p} \in \mathcal{G}^{\mathbf{2}} \mid \mathbf{p} \neq \mathbf{p}^\dagger \in \mathcal{G}\}. \tag{8.5}$$

Let $t \in \mathbb{C}$ be a primitive $2(N+L)$-th root of 1. Let ϵ be either 1 or -1 and ζ a nonzero parameter. We define a star-triangular face model $(\mathcal{G}, {}_{\mathcal{N},,\epsilon}) = (\mathcal{G}_{\mathcal{N},\mathcal{L}}, {}_{\mathcal{N},,\epsilon,\zeta})$ by setting

$$w_{N,t,\epsilon}\begin{bmatrix} \lambda & \lambda + \hat{i} \\ \lambda + \hat{i} & \lambda + \hat{i} + \hat{j} \end{bmatrix} = -\zeta^{-1} t^{-d(\mathbf{p})} \frac{1}{[d(\mathbf{p})]}, \tag{8.6}$$

$$w_{N,t,\epsilon}\begin{bmatrix} \lambda & \lambda + \hat{i} \\ \lambda + \hat{j} & \lambda + \hat{i} + \hat{j} \end{bmatrix} = \zeta^{-1} \epsilon \frac{[d(\mathbf{p}) - \mathbf{1}]}{[\mathbf{d}(\mathbf{p})]}, \tag{8.7}$$

$$w_{N,t,\epsilon}\begin{bmatrix} \lambda & \lambda + \hat{k} \\ \lambda + \hat{k} & \lambda + 2\hat{k} \end{bmatrix} = \zeta^{-1} t \tag{8.8}$$

for each $\mathbf{p} = (\lambda \,|\, \mathbf{i}, \mathbf{j}) \in \mathcal{G}[\searrow] \amalg \mathcal{G}[\downarrow]$ and $(\lambda \,|\, k, k) \in \mathcal{G}[\rightarrow]$, where $[n] = (t^n - t^{-n})/(t - t^{-1})$ for each $n \in \mathbb{Z}$. We call $(\mathcal{G}, {}_{\mathrm{N},,\epsilon})$ *$SU(N)_L$-SOS model* (without spectral parameter) [19]. Now the $SU(N)_L$-SOS algebra $\mathfrak{S}(A_{N-1}; t)_\epsilon$ is defined as the following quotient of the FRT construction $\mathfrak{A}(w_{N,t,\epsilon})$:

$$\mathfrak{S}(A_{N-1}; t)_\epsilon := \mathfrak{A}(w_{N,t,\epsilon})/(\det - 1), \tag{8.9}$$

where the group-like element $\det = \sum_{\lambda,\mu \in \mathcal{V}} \det \binom{\lambda}{\mu}$ of $\mathfrak{A}(w_{N,t,\epsilon})$ is defined by

$$\det \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \frac{D(\mu)}{D(\lambda)} \sum_{\mathbf{p} \in \mathcal{G}^{\mathcal{N}}_{\lambda\lambda}} (-\epsilon)^{\mathcal{L}(\mathbf{p}) + \mathcal{L}(\mathbf{q})} e \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix}. \tag{8.10}$$

Here $\mathbf{q}$ denotes an arbitrary element of $\mathcal{G}^{\mathcal{N}}_{\mu\mu}$, and $\mathcal{L} : \mathcal{G} \to \mathbb{Z}_{\geq \not{}}$ and $D(\lambda) \in \mathbb{C}$ are given by

$$\mathcal{L}(\lambda \,|\, , \dots,) = \mathrm{Card}\{(\mathrm{k}, \mathrm{l}) | 1 \leq \mathrm{k} < \mathrm{l} \leq \mathrm{N}, \mathrm{i_k} < \mathrm{i_l}\}, \tag{8.11}$$

$$D(\lambda) = \prod_{1 \leq i < j \leq N} \frac{[d(\lambda \,|\, i, j)]}{[d(0 \,|\, i, j)]} \quad (\lambda \in \mathcal{V}). \tag{8.12}$$

The canonical braiding of $\mathfrak{A}(\mathfrak{w}_{\mathfrak{N},\mathfrak{t},\epsilon,\zeta})$ induces the braiding $\mathcal{R}^+_\zeta$ of $\mathfrak{S}(A_{N-1}; t)_\epsilon$ if and only if ζ satisfies $\zeta^N = \epsilon^{N-1} t$. The face algebra $\mathfrak{S}(A_{N-1}; t)_\epsilon$ has an antipode whose square is given by

$$S^2 \left(e \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} \right) = \frac{D(\mathfrak{r}(\mathbf{p}))\mathbf{D}(\mathfrak{s}(\mathbf{q}))}{\mathbf{D}(\mathfrak{s}(\mathbf{p}))\mathbf{D}(\mathfrak{r}(\mathbf{q}))} e \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} \quad (\mathbf{p}, \mathbf{q} \in \mathcal{G}, \ \geq). \tag{8.13}$$

The $\mathfrak{S}$-comodule $\mathbb{K}\mathcal{G}$ is irreducible, while the $\mathfrak{S}$-comodule $\mathbb{K}\mathcal{G}$ has the irreducible decomposition:

$$\mathbb{K}\mathcal{G} = \mathrm{Ker}(\mathrm{w}_{\mathrm{N},\mathrm{t},\epsilon,\zeta} - \zeta^{-1}\mathrm{t}) \oplus \mathrm{Ker}(\mathrm{w}_{\mathrm{N},\mathrm{t},\epsilon,\zeta} + \zeta^{-1}\mathrm{t}^{-1}). \tag{8.14}$$

The proof of the following result is quite similar to that of Theorem 7.5 and Takeuchi [35] Lemma 2.4, hence we omit it.

Theorem 8.1. *Any braiding of $\mathfrak{S}(A_{N-1}; t)_\epsilon$ is either of the form $\mathcal{R}^+_\zeta$ or of the form $(\mathcal{R}^-_\zeta)_{21}$, where ζ denotes a solution of $\zeta^N = \epsilon^{N-1} t$.*

Similarly to Corollary 7.6, we can rewrite the result above in terms of the $\mathbb{Z}/\mathbb{N}\mathbb{Z}$-grading of $\mathfrak{S}(A_{N-1}; t)_\epsilon$ induced by $\mathfrak{A}(\mathfrak{w}_{\mathfrak{N},\mathfrak{t},\epsilon}) = \bigoplus_n \mathfrak{A}_\mathfrak{n}(\mathfrak{w}_{\mathfrak{N},\mathfrak{t},\epsilon})$.

Theorem 8.2. *When N is odd, $\mathfrak{S}(A_{N-1}; t)_\epsilon$ has exactly one ribbon functional. The corresponding modified ribbon functional $\mathcal{M}_+$ is given by*

$$\mathcal{M}_+ = \sum_{\|, \updownarrow \in \mathcal{V}} \frac{\mathcal{D}(\updownarrow)}{\mathcal{D}(\|)} \mathring{\rceil}_{\|} \rceil_{\updownarrow}. \tag{8.15}$$

When N is even, $\mathfrak{S}(A_{N-1}; t)_\epsilon$ has exactly two ribbon functionals. The corresponding modified ribbon functionals $\mathcal{M}_\pm$ are given by (8.15) *and*

$$\left\langle \mathcal{M}_-, \rceil \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} \right\rangle = \delta_{\mathbf{pq}} (-1)^m \frac{D(\mathfrak{r}(\mathbf{p}))}{\mathbf{D}(\mathfrak{s}(\mathbf{p}))} \quad (\mathbf{p}, \mathbf{q} \in \mathcal{G}, \ \geq). \tag{8.16}$$

Proof. By Theorem 6.7, we have $\mathrm{MRib}(\mathrm{Hc}(\mathfrak{A}(\mathrm{w}_{\mathrm{N},\mathrm{t},\epsilon}))) = \{\Phi(\pm M)\}$, where $M \in GL(\mathbb{K}\mathcal{G})$ is given by $M\mathbf{p} = \mathbf{D}(\mathfrak{r}(\mathbf{p}))\mathbf{D}(\mathfrak{s}(\mathbf{p}))^{-\mathbf{1}}\mathbf{p}$ ($\mathbf{p} \in \mathcal{G}$). Since

$$\begin{aligned} \left\langle \Phi(\pm M), e \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} \right\rangle &= \left(\pm \delta_{\mathbf{p_1 q_1}} \frac{D(\mathfrak{r}(\mathbf{p_1}))}{\mathbf{D}(\mathfrak{s}(\mathbf{p_1}))} \right) \cdots \left(\pm \delta_{\mathbf{p_m q_m}} \frac{D(\mathfrak{r}(\mathbf{p_m}))}{\mathbf{D}(\mathfrak{s}(\mathbf{p_m}))} \right) \\ &= (\pm 1)^m \delta_{\mathbf{pq}} \frac{D(\mathfrak{r}(\mathbf{p}))}{\mathbf{D}(\mathfrak{s}(\mathbf{p}))} \end{aligned}$$

for each $\mathbf{p} = (\mathbf{p_1} \dots \mathbf{p_m})$, $\mathbf{q} = (\mathbf{q_1}, \dots \mathbf{q_m}) \in \mathcal{G}$, we have

$$\langle \Phi(\pm M), \det - 1 \rangle = \mathrm{Card}(\mathcal{V})((\pm)^{\mathcal{N}} -). \tag{8.17}$$

By Proposition 6.8, this proves the assertion. □

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